4. On suppose qu'à partir du point (1,1) la variation relative de x est de 2% et celle de f est (-3%). Quelle variation relative de y a pu donner ce résultat (on fera un calcul approché).
5. f est elle convexe ou concave sur son domaine.

Exercice n°6 :

Soit f et g les fonctions définies sur l'ensemble $D = \{(x, y) \in IR^2 \; telsque \; x > 0 \; et \; y > 0\}$ par :

$$f(x,y) = xy^{\frac{1}{3}} \quad \text{et} \quad g(x,y) = x^2 . y^{\frac{-1}{3}}$$

1. a) Justifier que f et g sont de classe C^1 sur D.

   b) Déterminer et représenter (sommairement) les courbes de niveau 1 de f et g.
2. a) Montrer que f et g sont homogènes et préciser leurs degrés d'homogénéité.

   b) On suppose que f augmente de 4% et g diminue de 1%. Calculer les variations relatives $\frac{\Delta x}{x}$ et $\frac{\Delta y}{y}$ des variables x et y.
3. on suppose $(x_0, y_0) = (1, 8)$.

   a) Ecrire les développements limités de f et g à l'ordre 1 au voisinage de $(1, 1)$.

   b) En utilisant des valeurs approchées pour les variations absolues de f et g. Calculer les accroissements qu'il faut donner à x et y à partir de $(x_0 = 1)$ et $(y_0 = 8)$ pour que f augmente de 0.03 et g reste constante.

Exercice n°7 :

Soient $f(x, y) = xe^y + ye^x$; $g(x, y) = lny. e^{x-2} + x + y$

1. a) Montrer que l'équation $f(x, y) = 0$ permet de définir au voisinage de (0,0) une fonction implicite φ.

   b) Calculer $\varphi'(0)$ et $\varphi''(0)$.

   c) Donner le développement limité de φ à l'ordre 2 au voisinage de 0 et déduire la tangente (T_0) et sa position avec la courbe de φ.
2. On se place au point (a, a) où $a \in IR_+^*$. On suppose les variables x et y augmentent toutes les deux de 5%. En utilisant un calcul approché, déterminer a pour que f augmente de 10%.
3. On se place au point (2, 1).

   a) Sachant que x diminue de 3%. Quelle doit être la variation relative de y pour que g augmente de 2% (on utilisera un calcul approché).

   b) Donner le développement de l'ordre 2 de g au point (2, 1) puis déduire l'équation du plan tangente $(\mathcal{C}_g)$ au point (2, 1, 3) et sa position avec $(\mathcal{C}_g)$.

   c) Montrer que l'équation $g(x, y) = 3$ permet de définir au voisinage de (2, 1) une fonction implicite Ψ et donner son développement limité à l'ordre 2 au voisinage de 2 ainsi que la tangente (T_2) à la courbe Ψ et sa position avec la courbe de Ψ.

FIN

1

Série 2

Exercice 1 :

a/ $f(x,y) = \sqrt{\ln(xy) - 1}$

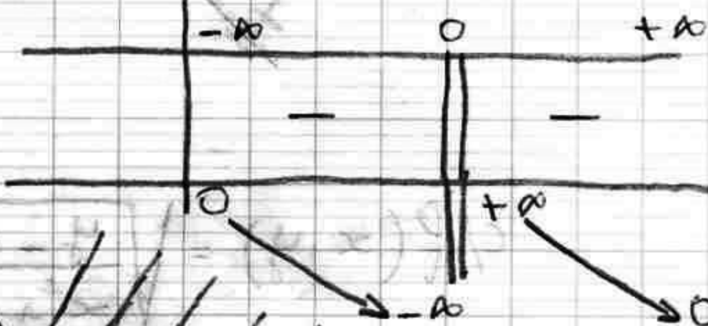
$D_f = \{(x,y) \in \mathbb{R}^2 / \ln(xy) - 1 \geqslant 0 \text{ et } xy > 0$

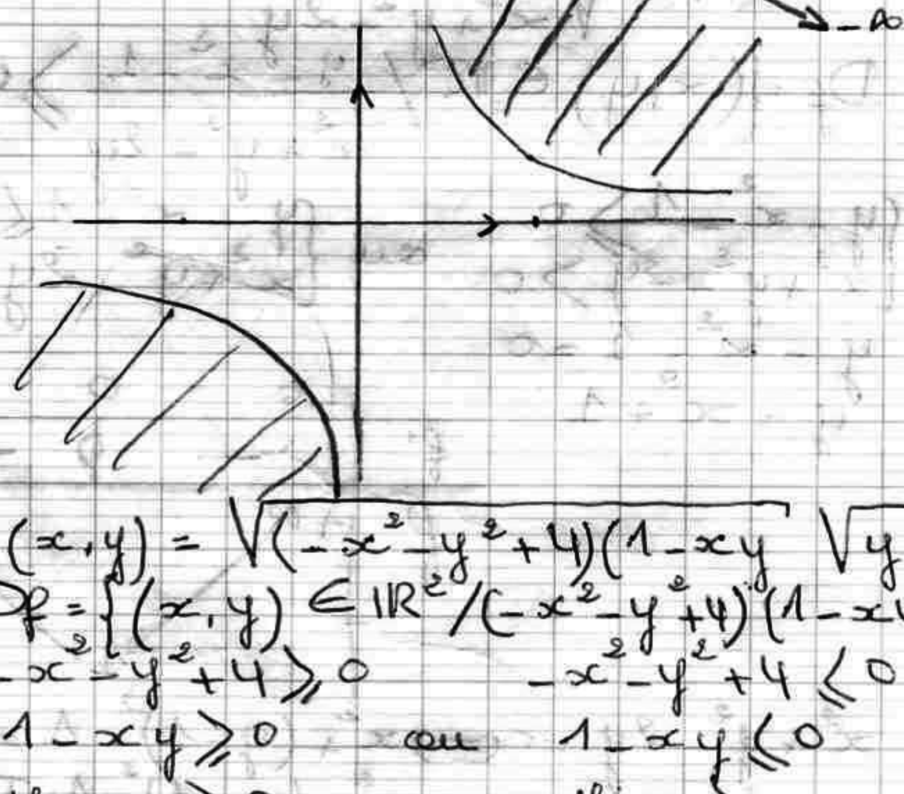
$\begin{cases} x > 0 \\ y > 0 \\ \ln(xy) - 1 \geqslant 0 \end{cases}$ ou $\begin{cases} x < 0 \\ y < 0 \\ \ln(xy) - 1 \geqslant 0 \end{cases}$

$\ln(xy) - 1 = 0$

$xy = e$

$y = \dfrac{e}{x}$





b/ $f(x,y) = \sqrt{(-x^2 - y^2 + 4)(1 - xy)}\ \sqrt{y - x}$

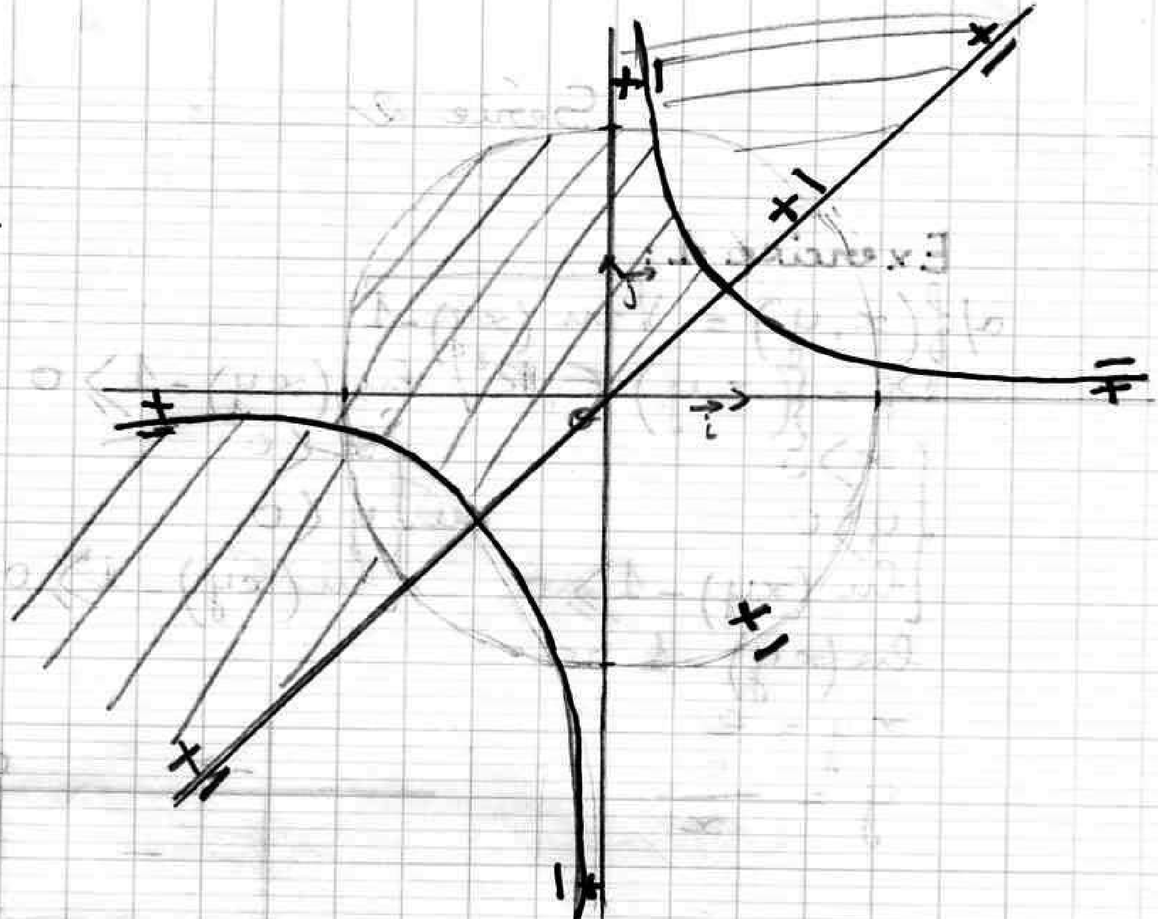
$D_f = \{(x,y) \in \mathbb{R}^2 / (-x^2 - y^2 + 4)(1 - xy) \geqslant 0 \text{ et } y - x \geqslant$

$\begin{cases} -x^2 - y^2 + 4 \geqslant 0 \\ 1 - xy \geqslant 0 \\ y - x \geqslant 0 \end{cases}$ ou $\begin{cases} -x^2 - y^2 + 4 \leqslant 0 \\ 1 - xy \leqslant 0 \\ y - x \geqslant 0 \end{cases}$

$x^2 + y^2 = 4 \ ; \ y = \dfrac{1}{x}$

$\Rightarrow$

2



c/ $f(x,y) = \sqrt{\dfrac{y-x^2-1}{x^2+y^2-2y}}$

$D_f = \left\{(x,y) \in \mathbb{R}^2 / \dfrac{y-x^2-1}{x^2+y^2-2y} \geq 0 \text{ et } x^2+y^2-2y \neq\right.$

$\begin{cases} y-x^2-1 \geq 0 \\ x^2+y^2-2y > 0 \end{cases}$ ou $\begin{cases} y-x^2-1 \leq 0 \\ x^2+y^2-2y < 0 \end{cases}$

$y-x^2-1=0$

$y=x^2+1$

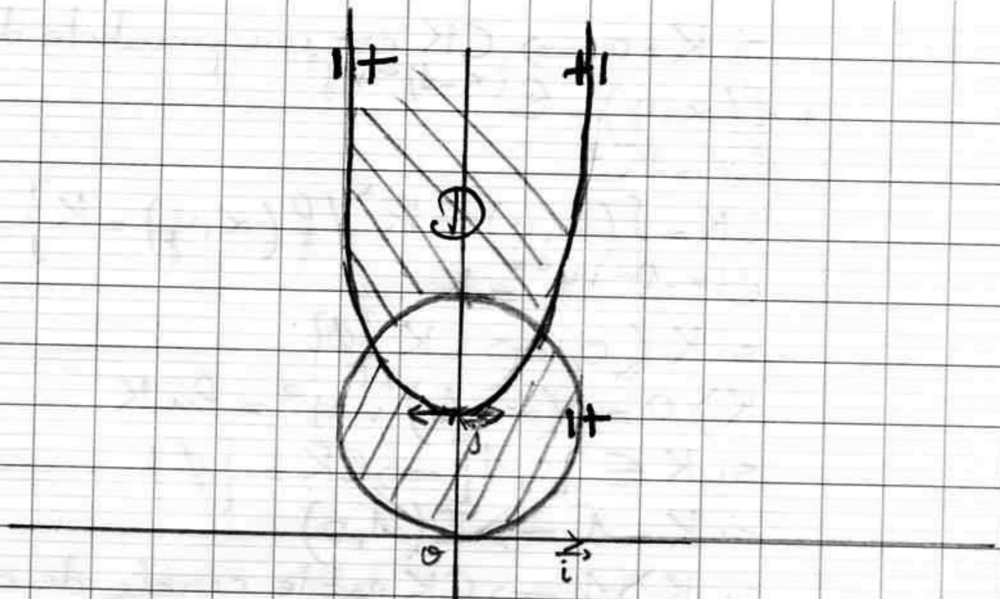
	$-\infty$		0		$+\infty$
f'		$-$	0	$+$	
f	$+\infty$	$\searrow$	1	$\nearrow$	$+\infty$

$x^2+y^2-2y=0 \Leftrightarrow x^2+(y-1)^2-1=0$

$x^2+(y-1)^2=1;\ I(0,1)\ R=1$

$\Rightarrow$

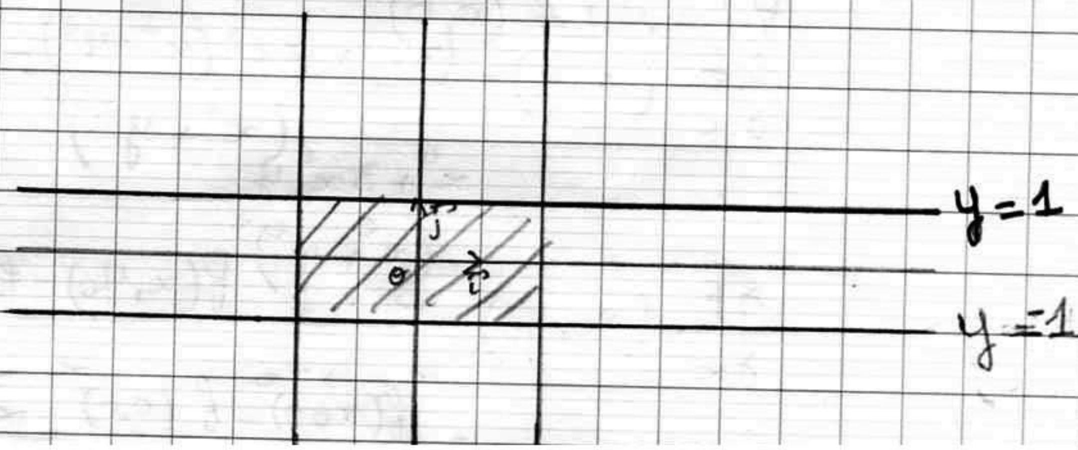
3)



d/ $f(x,y) = \sqrt{4 - x^2} + \sqrt{1 - y^2}$

$D_f = \{(x,y) \in \mathbb{R}^2 / 4 - x^2 \geqslant 0 \text{ et } 1 - y^2 \geqslant 0\}$

$D_f = [-2, 2] \times [-1, 1]$



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Exercice 2:

a/ * $f(x,y) = \dfrac{ay - 2}{x^2}$ $(a \in \mathbb{R}^*)$

$D_f = \mathbb{R}^* \times \mathbb{R}$

$C_K = \{(x,y) \in \mathbb{R}^2 / f(x,y) = K\}$

$f(x,y) = K \Longleftrightarrow \dfrac{ay - 2}{x^2} = K$ c'est à dire $y - 2 = x^2 . K$

$y = \dfrac{K}{a} . x^2 + \dfrac{2}{a}$

si $K = 0 \Longrightarrow C_0$ est la droite d'équation $y = \dfrac{2}{a}$

2

si $K=0 \Longrightarrow CK$ est une parabole de sommet $S(0, \frac{2}{a})$.

$* \; f(x,y) = e^{(x-1)^2+y^2}$

$Df = \mathbb{R}^2$

$CK = \{(x,y) \in \mathbb{R}^* / f(x,y) = K\}$

$e^{(x-1)^2+y^2} = K$

si $K \leq 0 \Longrightarrow CK \{\phi\}$

$K > 0 \Longrightarrow (x-1)^2 + y^2 = \ln K$

si $K \in]0,1[\Longrightarrow CK = \{\phi\}$

si $K = 1 \Longrightarrow I(1,0)$

si $K > 1 \Longrightarrow CK$ est le cercle de centre $I(1,0)$ et

$R = \sqrt{\ln K}$

1

Exercice 3:

$$\begin{cases} f(x,y) = \dfrac{x^3+x^2+y^2}{x^2+y^2} \text{ si } (x,y) \neq (0,0) \\ f(0,0) = 1 \end{cases}$$

$D_f = \mathbb{R}^2$

$\forall (x,y) \neq (0,0)$

$$\frac{\partial f}{\partial x}(x,y) = \frac{(3x^2+2x)(x^2+y^2) - 2x(x^3+x^2+y^2)}{(x^2+y^2)^2}$$

$$= \frac{x^4+3x^2y}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial x}(x_0,y_0) = \lim_{x \to x_0} \frac{f(x,y_0)-f(x_0,y_0)}{x-x_0}$$

$$= \frac{f(x,0)-f(0,0)}{x} = \frac{x+1-1}{x} = 1$$

$$\lim_{x \to 0} \frac{f(x,0)-f(0,0)}{x} = 1 = \frac{\partial f}{\partial x}(0,0)$$

$$\frac{f(0,y)-f(0,0)}{y} = \frac{1-1}{y} = 0$$

d'où $\displaystyle\lim_{y \to 0} \frac{f(0,y)-f(0,0)}{y} = 0 = \frac{\partial f}{\partial y}(0,0)$

Exercice 4 :

a/ $f(x,y) = e^{x-y} + x^2 - xy + y^2 + x + y$

$D_f = \mathbb{R}^2$

$$\frac{\partial f}{\partial x}(x,y) = e^{x-y} + 2x - y + 1$$

$$\frac{\partial f}{\partial x}(x,y) = -e^{x-y} - x + 2y + 1$$

$$\frac{\partial^2 f}{\partial x^2}(x,y) = e^{x-y} + 2$$

$$\frac{\partial^2 f}{\partial x y}(x,y) = -e^{x-y} - 1$$

$$\frac{\partial^2 f}{\partial y^2} = e^{x-y} + 2$$

b/ $f(x,y) = \dfrac{x^2+y^2}{x-y}$

$D_f = \{(x,y) \in \mathbb{R}^2 \setminus x - y \neq 0\}$

$$\frac{\partial f}{\partial x}(x,y) = \frac{2x(x-y) - (x^2+y^2)}{(x-y)^2}$$

$$= \frac{x^2 - 2xy - y^2}{(x-y)^2}$$

$$\frac{\partial f}{\partial x}(x,y) = \frac{2y(x-y) + x^2 + y^2}{(x-y)^2}$$

$$= \frac{y^2 + 2xy + x^2}{(x-y)^2}$$

$$\frac{\partial^2 f}{\partial x^2}(x,y) = \frac{(2x-2y)(x-y)^2 - 2(x-y)[x^2 - y^2 + 2xy}{(x-y)^2}$$

$$= \frac{(2x-2y)(x-y) - 2[x^2 - y^2 + 2xy]}{(x-y)^2}$$

$$= \frac{-8xy + 4y^2}{(x-y)^3}$$

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = \frac{(-2x-2y)(x-y)^2 + 2(x-y)[x^2 - 2xy - y}{(x-y)^4}$$

$$= \frac{(-2x-2y)(x-y)+2(x^2-2xy-y^2)}{(x-y)^3}$$

2

$$= \frac{-8xy-4y^2}{(x-y)^3} = \frac{\partial^2 f}{\partial y\, dx}$$

$$\frac{\partial^2 f}{\partial y^2}(x,y) = \frac{(-2y+2x)(x-y)^2+2(x-y)[-y^2+2xy+}{(x-y)^4}$$

$$= \frac{(-2y+2x)(x-y)+2(-y^2+2xy+x^2)}{(x-y)^3}$$

$$= \frac{4x^2}{(x-y)^3}$$

c/ $f(x,y) = \ln y\, e^{x-2} + x + y$

$D_f = \mathbb{R} \times]0,+\infty[$

$$\frac{\partial f}{\partial x}(x,y) = \ln y\, e^{x-2} + 1$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{e^{x-2}}{y} + 1$$

$$\frac{\partial^2 f}{\partial x^2}(x,y) = \ln y \cdot e^{x-2}$$

$$\frac{\partial^2 f}{\partial x\, dy}(x,y) = \frac{e^{x-2}}{y} = \frac{\partial^2 f}{\partial y \partial x}, \quad \frac{\partial^2 f}{\partial y^2}(x,y) = \frac{-e^{x-2}}{y^2}$$

d/ $f(x,y) = x \ln y - y \ln y$

$D_f =]0,+\infty[\times]0,+\infty[$

$$\frac{\partial f}{\partial x}(x,y) = \ln y - \frac{y}{x}, \quad \frac{\partial f}{\partial y}(x,y) = \frac{x}{y} - \ln x$$

$$\frac{\partial^2 f}{\partial x^2}(x,y) = \frac{y}{x^2}, \quad \frac{\partial^2 f}{\partial x \partial y}(x,y) = \frac{1}{y} - \frac{1}{x}$$

$$\frac{\partial^2 f}{\partial y^2}(x,y) = \frac{-x}{y^2}$$

e/ $f(x,y) = (x^2 + xy + y^2 + 1) - \ln(x+y)$

$D_f = \{(x,y) \ / x+y > 0\}$

$$\frac{\partial f}{\partial x}(x,y) = 2x + y - \frac{1}{x+y}$$

3

$$\frac{\partial f}{\partial y}(x,y) = x + 2y - \frac{1}{x+y}$$

$$\frac{\partial^2 f}{\partial x^2}(x,y) = 2 + \frac{1}{(x+y)^2}$$

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = 1 + \frac{1}{(x+y)^2}$$

$$\frac{\partial^2 f}{\partial y^2}(x,y) = 2 + \frac{1}{(x+y)^2}$$

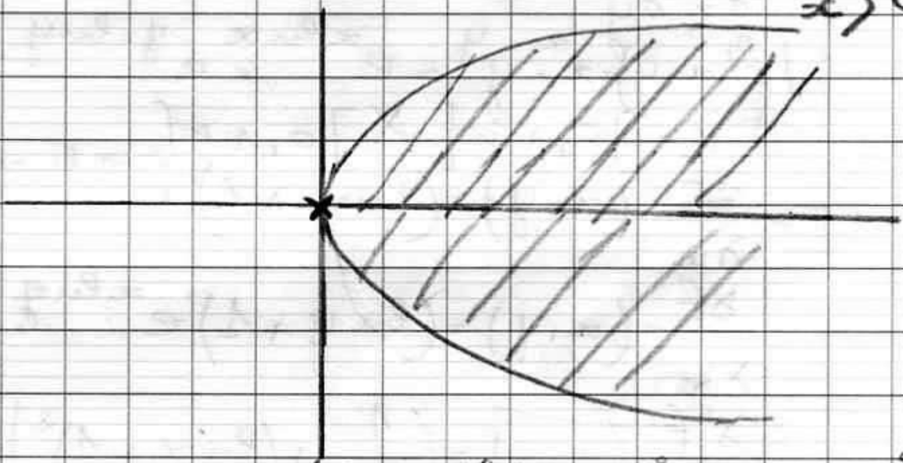
f/ $f(x,y) = \ln\left(\frac{x^2}{2x-y^2}\right)$

$D_f = \{(x,y) \in \mathbb{R}^2 / 2x - y^2 > 0 \text{ et } x \neq 0\}$

$2x - y^2 = 0$

$y^2 = 2x$, $y = \pm\sqrt{2x}$

$x > 0$



$$\frac{\partial f}{\partial x}(x,y) = \frac{2x(2x-y^2) - 2x^2}{(2x-y^2)^2} \cdot \frac{2x-y^2}{x^2}$$

$$= \frac{2x^2 - 2xy^2}{x^2(2x-y^2)} = \frac{2x - 2y^2}{x(2x-y^2)}$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{2yx^2}{(2x-y^2)^2} \cdot \frac{2x-y^2}{x^2}$$

$$= \frac{2y}{2x-y^2}$$

$$\frac{\partial^2 f}{\partial x^2}(x,y) = \frac{2x(2x-y^2) - 4x(2x-2y^2)}{x^2(2x-y^2)^2}$$

$$= \frac{-4x^2 + 6xy^2}{x^2(2x-y^2)^2} = \frac{-4x + 6y^2}{x(2x-y^2)^2}$$

4

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = \frac{-4y}{(2x-y^2)^2}$$

$$\frac{\partial^2 f}{\partial y^2}(x,y) = \frac{4x - 2y^2 + 4y^2}{(2x-y^2)^2} = \frac{4x+2y^2}{(2x-y^2)^2}$$

g/ $f(x,y) = x - y + \frac{1}{2}(e^{x-y} + e^{y-x})$

$D_f = \mathbb{R}^2$

$$\frac{\partial f}{\partial x}(x,y) = 1 + \frac{1}{2}(e^{x-y} - e^{y-x})$$

$$\frac{\partial f}{\partial y} = -1 + \frac{1}{2}(-e^{x-y} + e^{y-x})$$

$$\frac{\partial^2 f}{\partial x^2}(x,y) = \frac{1}{2}(e^{x-y} + e^{y-x})$$

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = \frac{1}{2}(e^{x-y} + e^{y-x})$$

h/ $f(x,y) = x^x \cdot y^y = e^{x \ln x} \cdot e^{y \cdot \ln y}$

$D_f =]0,+\infty[\times]0,+\infty[$

$$\frac{\partial f}{\partial x}(x,y) = (\ln x + 1)\, e^{x \ln x} \cdot e^{y \ln y}$$

$$\frac{\partial f}{\partial y}(x,y) = (\ln y + 1)\, e^{x \ln x} \cdot e^{y \ln y}$$

$$\frac{\partial^2 f}{\partial x^2}(x,y) = \left(\frac{1}{x} + (\ln x + 1)^2\right) e^{x \ln x} \cdot e^{y \ln y}$$

$$\frac{\partial^2 f}{\partial y^2}(x,y) = \left(\frac{1}{y} + (\ln y + 1)^2\right) e^{x \ln x} \cdot e^{y \ln y}$$

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = (\ln x + 1)(\ln y + 1) \cdot e^{x \ln x} \cdot e^{y \ln y}$$

Exercice 6 :

1/a/ $f(x,y) = x y^{1/3}$; $g(x,y) = x^2 y^{-1/3}$

$$\frac{\partial f(x,y)}{\partial x} = y^{1/3} \; ; \; \frac{\partial f(x,y)}{\partial y} = \frac{1}{3} y^{-2/3} x$$

$$\frac{\partial g(x,y)}{\partial x} = 2x y^{-1/3} \; ; \; \frac{\partial g(x,y)}{\partial y} = -\frac{1}{3} y^{-4/3} x^2$$

f et g sont continues sur $]0,+\infty[\times]0,+\infty[$ et les dérivées partielles sont continues

$\Longrightarrow$ f et g de C^1 sur $]0,+\infty[\times]0,+\infty[$

b/ $C_1 = \{(x,y) \in \mathbb{R}^2 \setminus f(x,y) = 1\}$

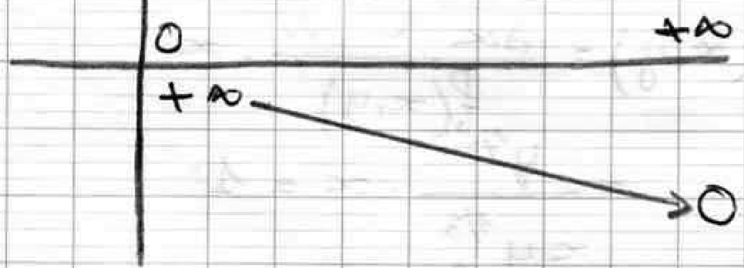
$f(x,y) = 1$ sig $x y^{1/3} = 1$

sig $y^{1/3} = \frac{1}{x}$

$\Longrightarrow y = \left(\frac{1}{x}\right)^3 = \frac{1}{x^3}$

Supposons que $f(x) = \frac{1}{x^3}$ $\left(\frac{1}{f}\right)' = \frac{-f'}{f^2}$

$f'(x) = \frac{-3x^2}{x^6} = \frac{-3}{x^4} < 0$

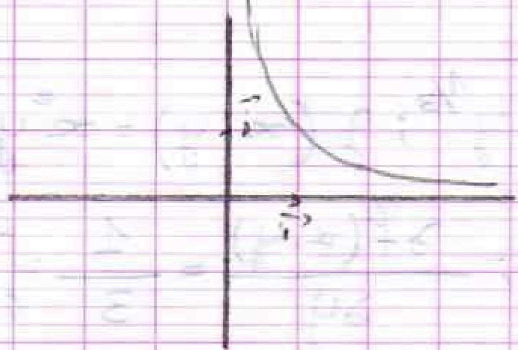


$C'_1 = \{(x,y) \in \mathbb{R}^2 \setminus g(x,y) = 1\}$

$g(x,y) = 1$ sig $x^2 y^{-\frac{1}{3}} = 1$

sig $y^{-1/3} = \frac{1}{x^2}$

sig $y = \left(\frac{1}{x^2}\right)^{-3} = \frac{1}{x^{-6}} = x^6$



2/a/ Soit $t > 0$

$$f(tx, ty) = tx(ty)^{\frac{1}{3}}$$
$$= tx\, t^{1/3} y^{1/3}$$
$$= t^{\frac{4}{3}} x y^{1/3}$$

$\Rightarrow f$ est homogène de degré $\frac{4}{3}$

$$g(tx, ty) = (tx)^2 (ty)^{-1/3}$$
$$= t^2 x^2 . t^{-1/3} y^{-1/3}$$
$$= t^{\frac{5}{3}} x^2 y^{-1/3}$$

$\Rightarrow g$ est homogène de degré $\frac{5}{3}$

b/ $\frac{\Delta f}{f} = 4\%$, $\frac{\Delta g}{g} = -1\%$

$$\frac{\Delta f}{f} = ef_{/x} \frac{\Delta x}{x} + ef_{/x} \frac{\Delta y}{y}$$

$$\frac{\Delta g}{g} = eg_{/x} \frac{\Delta x}{x} + eg_{/y} \frac{\Delta y}{y}$$

c/ $$ef_{/x}(x,y) = \frac{\frac{\partial f}{\partial x}(x,y)}{f(x,y)} . x$$
$$= \frac{y^{1/3}}{x y^{1/3}} . x = 1$$

$$ef_{/y}(x,y) = \frac{\frac{\partial f}{\partial y}(x,y) . y}{f(x,y)}$$
$$= \frac{\frac{1}{3} x y^{2/3}}{x^2 y^{-1/3}} . y = \frac{1}{3}$$

$$eg_{/x}(x,y) = \frac{\frac{\partial g}{\partial x}(x,y) . x}{g(x,y)} = \frac{2xy^{-1/3}}{x^2 y^{-1/3}} . x = 2$$

$$e_{g/y}(x,y) = \frac{\frac{\partial g}{\partial y}(x,y)\cdot y}{g(x,y)}$$

$$= \frac{-\frac{1}{3}\, y^{4/3} x^2}{x^2 y^{-1/3}} \cdot y = -\frac{1}{3}$$

$$4\% = \frac{\Delta x}{x} + \frac{1}{3} \cdot \frac{\Delta y}{y} \quad (1)$$

$$-1\% = \frac{2\Delta x}{x} - \frac{1}{3}\,\frac{\Delta y}{y} \quad (2)$$

$$(1) + (2) \Longrightarrow 3\% = \frac{3\Delta x}{x} \Longrightarrow \frac{\Delta x}{x} = 1\%$$

$$(2) \Longrightarrow -1 = 2 - \frac{1}{3}\,\frac{\Delta y}{y} \Longrightarrow \frac{\Delta y}{y} = 9\%$$

3/ $(x_0, y_0) = (1,8)$

a/ $DL_1(1,8)$

$\forall x,y \in \mathcal{V}(1,8)$

$$f(x,y) = f(1,8) + (x-1)\frac{\partial f}{\partial x}(1,8) + (y-8)\frac{\partial f}{\partial y}(1,8)$$

$$+\sqrt{(x-1)^2+(y-8)^2}\;\varepsilon(x,y)$$

$f(1,8) = 1 \cdot 8^{1/3} = 2$

$$\frac{\partial f(1,8)}{\partial y} = \frac{1}{3} \cdot 8^{-2/3} = \frac{1}{12}$$

$$\frac{\partial f}{\partial x}(1,8) = 8^{1/3} = 2$$

$$f(x,y) = 2 + 2(x-1) + \frac{1}{12}(y-8) + R$$

$z = 2 + 2(x-1) + \frac{1}{12}(y-8)$ est l'équation du plan tengant à la surface de la courbe au point $(1,8,2)$

$\forall x,y \in$ au $\mathcal{V}(1,8)$

$$g(x,y) = g(1,8) + (x-1)\frac{\partial g}{\partial x}(1,8) + (y-8)\frac{\partial g(1,8)}{\partial y} +$$

4

$$\sqrt{(x-1)^2+(y-8)^2}\ \varepsilon(x,y)$$

$$g(1,8) = 1.\ 8^{-1/3} = \frac{1}{2}$$

$$\frac{\partial g}{\partial x}(1,8) = 2.8^{-1/3} = 1$$

$$\frac{\partial g(1,8)}{\partial y} = -\frac{1}{3}.8^{-1/3} = -\frac{1}{48}$$

$$g(x,y) = \frac{1}{2} + (x-1)\frac{1}{48}(y-8) + R$$

b/ $\Delta f = 0{,}03$, $\Delta g = 0$

$$\begin{cases} \Delta f = \frac{\partial f}{\partial x}(1,8)\cdot\Delta x + \frac{\partial f}{\partial y}(1,8)\cdot\Delta y \\ \Delta g = \frac{\partial g}{\partial x}(1,8)\,\Delta x + \frac{\partial g}{\partial y}(1,8).\,\Delta y \end{cases}$$

$$\begin{cases} 0{,}03 = 2\Delta x + \frac{1}{12}\Delta y \quad (1) \\ 0 = \Delta x - \frac{1}{48}\Delta y \quad (2) \end{cases}$$

$$\begin{cases} 0{,}03 = \frac{3}{24}\Delta y \Rightarrow \Delta y = 0{,}24 \\ \Delta x = \frac{1}{48}\times 0{,}24 = 0{,}005 \end{cases}$$

Exercice 7:

$$f(x,y) = x e^y + y e^x$$

1/ a/ Calculons $f(0,0)$

(1) $f(0,0) = 0.1 + 1.0 = 0$

(2) $$\left.\begin{array}{l} \frac{\partial f}{\partial x}(x,y) = e^y + y e^x \\ \frac{\partial f}{\partial y}(x,y) = x e^y + e^x \end{array}\right\} \Rightarrow f \text{ de classe } C^1 \text{ au } \mathcal{V}(0,0)$$

1

$y(x,$

$)$

Exercice 7:

$f(x,y) = x e^{y} + y e^{x}$

1/ a/ Calculons $f(0,0)$

① $f(0,0) = 0.1 + 1.0 = 0$

② $$\left.\begin{aligned} \frac{\partial f}{\partial x}(x,y) &= e^{y} + y e^{x} \\ \frac{\partial f}{\partial y}(x,y) &= x e^{y} + e^{x} \end{aligned}\right\} \Rightarrow f \text{ de classe } C^1 \text{ au } \mathcal{V}(0,0)$$

③ $\frac{\partial f}{\partial y}(0,0) = 1 \neq 0$

① + ② + ③ $\Longrightarrow$ il $\exists$ une fonction implicite φ définie en 0 tel que $\varphi(0) = 0$ et $\forall x \in \mathcal{V}(0), f(x, \varphi(x))$

b/ $\varphi'(x) = \dfrac{-\frac{\partial f}{\partial x}(x, \varphi(x))}{\frac{\partial f}{\partial y}(x, \varphi(x))}$

$$\varphi'(0) = \frac{-\frac{\partial f}{\partial x}(0,0)}{\frac{\partial f}{\partial y}(0,0)} = \frac{-1}{1} = -1$$

Pour tout $x \in \mathcal{V}(0)$

$$\varphi'(x) = -\frac{e^{\varphi(x)} + \varphi(x)\, e^{x}}{x e^{\varphi(x)} + e^{x}}$$

$$\varphi'(x) = \frac{(\varphi'(x)e^{\varphi(x)} + \varphi'(x)e^{x} + e^{x}\varphi(x)).(x e^{\varphi(x)} + e^{x}) - [e^{\varphi(x)} + \varphi'(x)e^{\varphi(x)}.x + e^{x})(e^{\varphi(x)} + \varphi(x)e^{x})]}{(x e^{\varphi(x)} + e^{x})^2}$$

$\Longrightarrow \varphi''(x) = -(-2-2) = 4$

$\forall x \in \mathcal{V}(0)$

$$\varphi(x) = \varphi(0) + x\varphi'(0) + \frac{x^2}{2}.f''(0) + \varepsilon(x).x^2$$
$$= -x + 2x^2 + \varepsilon(x)x^2$$

$\Delta : y = -x$ est la tangente en 0 de $\varphi \Longrightarrow f(x) - y = 2x^2$;
$\Longrightarrow$ au $\mathcal{V}(0)$ $\varphi / \hat{\Delta}$

$$\frac{\partial f}{\partial x}(x,y) = e^{y} + y e^{x}$$

$$\frac{\partial f}{\partial y}(x,y) = x e^{y} + e^{x}$$

2/ au $\mathcal{V}(a,a)$; $\begin{cases} \frac{\Delta x}{x} = \frac{\Delta y}{y} = 5\% \\ \frac{\Delta f}{f} = 10\% \end{cases}$

On a : $\frac{\Delta f}{f} \simeq e f_{/x}(a,a).\frac{\Delta x}{x} + e f_{/y}(a,a).\frac{\Delta y}{y}$

$$e f_{/x}(a,a) = \frac{\frac{\partial f}{\partial x}(a,a)}{f(a,a)}.a = \frac{e^{a}(1+a)}{2a\, e^{a}}$$

$$= \frac{1+a}{2}$$

$$ef_{/y}(a,a) = \frac{\frac{\partial f}{\partial y}(a,a)}{f(a,a)}$$

$$= \frac{e^a(1+a)}{2ae^a} \cdot a = \frac{1+a}{2}$$

$$10\% = \frac{1+a}{2} \cdot 5\% + \frac{1+a}{2} \cdot 5\%$$

$$2 = 1 + a \Longrightarrow \boxed{a = 1}$$

3/ $g(x,y) = \ln y \cdot e^{x-2} + x + y$; $D_g = \mathbb{R} \times]0, +\infty[$

a/ $V(2,1)$; $\begin{cases} \frac{\Delta x}{x} = -3\% \\ \frac{\Delta g}{g} = 2\% \end{cases}$, $\frac{\Delta y}{y} = ?$

$$\frac{\partial g(x,y)}{\partial x} = \ln y\, e^{x-2} + 1 \; ; \; \frac{\partial g(2,1)}{\partial x} = 1$$

$$\frac{\partial g(x,y)}{\partial y} = \frac{e^{x-2}}{y} + 1 \; ; \; \frac{\partial g(2,1)}{\partial y} = 2$$

$g(2,1) = 3$

On a : $\frac{\Delta g}{g} = eg_{/x}(2,1) \cdot \frac{\Delta x}{x} + eg_{/y}(2,1) \cdot \frac{\Delta y}{y}$

$$eg_{/x}(2,1) = \frac{\frac{\partial g}{\partial x}(2,1)}{g(2,1)} \cdot 2 = \frac{2}{3}$$

$$eg_{/y}(2,1) = \frac{\frac{\partial g}{\partial y}(2,1)}{g(2,1)} = \frac{2}{3}$$

$$2\% = -\frac{2}{3} \cdot 3\% + \frac{2}{3} \frac{\Delta y}{y}$$

$\Longrightarrow \frac{\Delta y}{y} = 4\% \times \frac{3}{2} = 6\% \Longrightarrow y$ augmente de 6 %

b/ $\frac{\partial^2 g(x,y)}{\partial x^2} = \ln y\, e^{x-2}$; $\frac{\partial^2 g(2,1)}{\partial^2 x} = 0$

$$\frac{\partial^2 g}{\partial x y} = \frac{e^{x-2}}{y} \; ; \; \frac{\partial^2 g(2,1)}{\partial x \partial y} = 1 = \frac{\partial^2 g}{\partial y \partial x}(2,1)$$

$$\frac{\partial^2 g}{\partial^2 y} = -\frac{e^{x-2}}{y^2} \; ; \; \frac{\partial^2 g(2,1)}{\partial^2 y} = -1$$

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$\forall x, y \in \mathcal{V}(2,1)$

$$g(x,y) = g(2,1) + (x-2) \cdot \frac{\partial g(2,1)}{\partial x} + (y-1)\frac{\partial g(2,}{\partial y}$$

$$+ \frac{1}{2}\left[(x-2)^2 \frac{\partial^2 g(2,1)}{\partial^2 x} + (y-1)^2 \frac{\partial^2 g(2,1)}{\partial^2 y}\right.$$

$$\left. + 2\left((x-2)(y-1)\frac{\partial^2 g(2,1)}{\partial x \partial y}\right] + \Sigma(x,y) \times\right.$$

$$\left[(x-2)^2 + (y-1)^2\right]$$

$$g(x,y) = 3 + (x-2) + 2(y-1) + \frac{1}{2}\left(-(y-1)^2 + 2(x-\right.$$

$$\times (y-1) + R$$

$z = 3 + (x-2) + 2(y-1)$ est l'équation du plan tengant à la surface de la courbe en $(2,1,3)$

- Cherchons $Hf(2,1)$

$Hf(2,1)\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$; $\det Hf(2,1) = -1 < 0$

$\Rightarrow$ La surface de $\mathcal{C}_g$ traverse son plan tengant en $(2,1,3)$

c/ * On a $g(2,1) = 3$

* g est de classe C^1 au $\mathcal{V}(2,1) \Rightarrow$ Il $\exists$ une fonction implicite Ψ définie au $\mathcal{V}(2)$ vérifiant $\Psi(2) = 1$

* $\frac{\partial g(2,1)}{\partial y} = 2 \neq 0$

$\forall x \in \mathcal{V}(2)$; $g(x, \Psi(x)) = 3$

$$\Psi'(2) = \frac{-\frac{\partial g(2,1)}{\partial x}}{\frac{\partial g(2,1)}{\partial y}} = -\frac{1}{2}$$

$$\Psi'(x) = \frac{\ln \Psi(x)\, e^{x-2} + 1}{\frac{e^{x-2}}{\Psi(x)} + 1}$$

$$= -\frac{\Psi(x)\left[\ln \Psi(x) e^{x-2} + 1\right]}{e^{x-2} + \Psi(x)}$$

$$\Psi''(x) = \frac{\left[\Psi'(x)(\ln \Psi(x) e^{x-2} + 1) + \Psi(x) \times \left(\frac{\Psi'(x)}{\Psi(x)} \cdot e^{x-2} + \ln \Psi(x) e^{x-2}\right]\cdot(e^{x-2} + \Psi\right.}{}$$

$$\frac{-(e^{x-2} + \Psi'(x))(\ln \Psi(x) e^{x-2} + 1) \cdot \Psi(}{(e^{x-2} + \Psi(x))^2}$$

$$\Psi''(2) = \frac{-(-\frac{1}{2} - 1/2) \cdot 2 - 1/2}{4} = \frac{5}{8}$$

Pour tout $x \in V(2)$

$$\Psi(x) = \Psi(2) + (x-2)\Psi'(2) + \frac{(x-2)^2}{2} \cdot \Psi''(2) +$$
$$\varepsilon(x)(x-2)^2 = 1 - \frac{(x-2)}{2} + \frac{5^2}{16}(x-2)^2 + R$$

$$y = 1 - \frac{(x-2)}{2} \text{ au } V(2)$$

$$\Rightarrow \Psi(x) - y = \frac{5}{16}(x+2)^2 \ngeq 0 \ \forall x \in V($$

$\Rightarrow$ Pour $x \in V(2)$ $\mathcal{C}_\Psi / T_2$