EXERCICE 1 :

on donne les matrices suivantes

$$A=\begin{pmatrix} 2 & 1 & 0 \\ -1 & 0 & 4 \\ 1 & 1 & 1 \end{pmatrix} \quad B=\begin{pmatrix} 3 & 0 & 1 \\ -1 & 2 & -1 \\ -2 & 0 & 0 \end{pmatrix} \quad C=\begin{pmatrix} 2 & 0 & 0 \\ -3 & -1 & -1 \\ 3 & 1 & -3 \end{pmatrix}$$

Pour chaque matrice

a) Déterminer les valeurs propres

b) Déterminer les sous espaces propres

c) Dire si la matrice est diagonalisable ou Non?

d) si la matrice est diagonalisable donner la matrice de passage P et la matrice $D=P^{-1}AP$

Solution

1/ a) $$P_A(\lambda)=\det(A-\lambda I_3)=\begin{array}{c} \begin{matrix} C_1 & C_2 & C_3 \end{matrix} \\ \begin{vmatrix} 2-\lambda & 1 & 0 \\ -1 & -\lambda & 4 \\ 1 & 1 & 1-\lambda \end{vmatrix} \end{array} = \begin{array}{c} \begin{matrix} C_1-C_2 & C_2 & C_3 \end{matrix} \\ \begin{vmatrix} 1-\lambda & 1 & 0 \\ -1+\lambda & -\lambda & 4 \\ 0 & 1 & 1-\lambda \end{vmatrix} \end{array}$$

$$P_A(\lambda)=(1-\lambda)\begin{array}{c} \begin{matrix} C_1 & C_2 & C_3 \end{matrix} \\ \begin{vmatrix} 1 & 1 & 0 \\ -1 & -\lambda & 4 \\ 0 & 1 & 1-\lambda \end{vmatrix} \end{array} = (1-\lambda)\begin{array}{c} \begin{matrix} C_1 & C_2-C_1 & C_3 \end{matrix} \\ \begin{vmatrix} 1 & 0 & 0 \\ -1 & 1-\lambda & 4 \\ 0 & 1 & 1-\lambda \end{vmatrix} \end{array}$$

$$\begin{aligned} P_A(\lambda)=(1-\lambda)\begin{vmatrix} 1-\lambda & 4 \\ 1 & 1-\lambda \end{vmatrix} &= (1-\lambda)\left[(1-\lambda)^2-4\right] \\ &= (1-\lambda)(1-\lambda-2)(1-\lambda+2) \\ &= (1-\lambda)(-\lambda-1)(-\lambda+3) \\ &= (\lambda-1)(\lambda+1)(3-\lambda) \end{aligned}$$

D'où Spectre $A=\{-1,1,3\}$

Les valeurs propres de A sont $\lambda_1=-1$ vp simple
$\lambda_2=1$ vp simple
$\lambda_3=3$ vp simple.

b) $E_{-1}=\{X\in\mathbb{R}^3 / AX=-X\}$

$$AX+X=0$$
$$(A+I_3)X=0$$
$$\begin{pmatrix} 3 & 1 & 0 \\ -1 & 1 & 4 \\ 1 & 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 3x+y=0 \\ -x+y+4z=0 \\ x+y+2z=0 \end{cases} \quad \begin{cases} y=-3x \\ -x-3x+4z=0 \quad z=x \end{cases}$$

D'au

$$\overset{X}{\begin{pmatrix} x \\ y \\ z \end{pmatrix}} = \overset{X}{\begin{pmatrix} x \\ -3x \\ x \end{pmatrix}} = x \overset{U}{\begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}}$$

$E_{-1} = \text{Vect}(U) \quad \dim E_{-1} = 1$

$E_1 = \{ X \in \mathbb{R}^3 / AX = X \}$

$$(A-I)X = 0$$

$$\begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & 4 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} x+y=0 \\ -x-y+4z=0 \\ x+y=0 \end{cases} \quad \begin{cases} y=-x \\ z=0 \end{cases}$$

$$\overset{X}{\begin{pmatrix} x \\ y \\ z \end{pmatrix}} = \overset{X}{\begin{pmatrix} x \\ -x \\ 0 \end{pmatrix}} = x \overset{V}{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}} \quad E_1 = \text{Vect}(V) \quad \dim E_1 = 1$$

$E_3 = \{ X \in \mathbb{R}^3 / AX = 3X \}$

$$AX - 3X = 0$$

$$(A - 3I_3) X = 0$$

$$\begin{pmatrix} -1 & 1 & 0 \\ -1 & -3 & 4 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -x+y=0 \\ -x-3y+4z=0 \\ x+y-2z=0 \end{cases} \quad \begin{cases} y=x \\ z=x \end{cases}$$

$$X \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ x \\ x \end{pmatrix} = x \overset{W}{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}} \quad E_3 = \text{Vect}(W) \quad \dim E_3 = 1$$

c) La matrice A est diagonalisable

car A admet Trois valeurs propres réelles distinctes

ou Bien $\dim E_{-1} + \dim E_1 + \dim E_3 = \dim \mathbb{R}^3$

d) La matrice de passage

$$P = \begin{array}{c} \begin{array}{ccc} u & v & w \end{array} \\ \begin{pmatrix} 1 & 1 & 1 \\ -3 & -1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \end{array} \begin{array}{c} e_1 \\ e_2 \\ e_3 \end{array}$$

$\det P \neq 0$ P est inversible

$$P^{-1}AP = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

2°/ a) $B = \begin{pmatrix} 3 & 0 & 1 \\ -1 & 2 & -1 \\ -2 & 0 & 0 \end{pmatrix}$

$$P_B(\lambda) = \det(B - \lambda I_3) = \begin{vmatrix} 3-\lambda & 0 & 1 \\ -1 & 2-\lambda & -1 \\ -2 & 0 & -\lambda \end{vmatrix}$$

$$P_B(\lambda) = (2-\lambda) \begin{vmatrix} 3-\lambda & 1 \\ -2 & -\lambda \end{vmatrix} = (2-\lambda)\left[-\lambda(3-\lambda)+2\right]$$

$$= (2-\lambda)(\lambda^2 - 3\lambda + 2)$$

$$= (2-\lambda)(\lambda-1)(\lambda-2)$$

$$P_B(\lambda) = (\lambda-2)^2(1-\lambda)$$

Les valeurs propres de B sont les solutions de $P_B(\lambda) = 0$

$\lambda_1 = 2$ valeur propre d'ordre 2

$\lambda_2 = 1$ valeur propre d'ordre 1 (simple)

b) $E_2 = \{ X \in \mathbb{R}^3 / BX = 2X \}$

$$(B - 2I_3)X = 0$$

$$\begin{pmatrix} 1 & 0 & 1 \\ -1 & 0 & -1 \\ -2 & 0 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} x + z = 0 \\ -x - z = 0 \\ -2x - 2z = 0 \end{cases} \Leftrightarrow z = -x$$

$$\begin{array}{c} X \\ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \end{array} = \begin{pmatrix} x \\ y \\ -x \end{pmatrix} = x \begin{array}{c} U \\ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \end{array} + y \begin{array}{c} V \\ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \end{array}$$

$E_2 = \text{vect}(U, V)$

U non colinéaire à $V \Leftrightarrow (U, V)$ libre

donc (U, V) base de E_2 et $\dim E_2 = 2$

$$E_1 = \{ X \in \mathbb{R}^3 / BX = X \}$$

$$(B - I_3)X = 0$$

$$\begin{pmatrix} 2 & 0 & 1 \\ -1 & 1 & -1 \\ -2 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 2x + z = 0 \\ -x + y - z = 0 \\ -2x - z = 0 \end{cases} \qquad \begin{cases} z = -2x \\ y = -x \end{cases}$$

$$\begin{array}{c} X \\ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \end{array} = \begin{pmatrix} x \\ -x \\ -2x \end{pmatrix} = x \begin{array}{c} W \\ \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \end{array}$$

$E_1 = \text{vect}(W)$ $\dim E_1 = 1$

c) $\dim E_2 + \dim E_1 = 3 = \dim \mathbb{R}^3$

Donc B est diagonalisable

d) La matrice de passage $P = \begin{array}{c} \begin{array}{ccc} U & V & W \end{array} \\ \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ -1 & 0 & -2 \end{pmatrix} \end{array} \begin{array}{c} e_1 \\ e_2 \\ e_3 \end{array}$

$\det P \neq 0$ $\quad P^{-1}BP = D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

<u>on remarque</u> $Tr(B) = 3 + 2 + 0 = 5$

$\lambda_1 + \lambda_2 + \lambda_3 = 2 + 2 + 1 = 5$

$\det B = 4$

$\lambda_1 \lambda_2 \lambda_3 = 2 \times 2 \times 1 = 4$

3°/ $C = \begin{pmatrix} 2 & 0 & 0 \\ -3 & -1 & -1 \\ 3 & 1 & -3 \end{pmatrix}$ $P_C(\lambda) = \det(C - \lambda I_3)$

a)

$$P_C(\lambda) = \begin{vmatrix} 2-\lambda & 0 & 0 \\ -3 & -1-\lambda & -1 \\ 3 & 1 & -3-\lambda \end{vmatrix}$$

$$= (2-\lambda) \begin{vmatrix} -1-\lambda & -1 \\ 1 & -3-\lambda \end{vmatrix}$$

$$= (2-\lambda)\left[(-1-\lambda)(-3-\lambda) + 1\right]$$

$$= (2-\lambda)(\lambda^2 + 4\lambda + 4)$$

$$= (2-\lambda)(\lambda+2)^2$$

Les valeurs propres de C sont les solutions de $P_C(\lambda) = 0$

$\lambda_1 = 2$ VP simple

$\lambda_2 = -2$ VP d'ordre 2

b) $E_2 = \{ X \in \mathbb{R}^3 / CX = 2X \}$

$(C - 2I_3)X = 0$

$$\begin{pmatrix} 0 & 0 & 0 \\ -3 & -3 & -1 \\ 3 & 1 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{matrix} L_1 \\ L_2 \end{matrix} \begin{cases} -3x - 3y - z = 0 \\ 3x + y - 5z = 0 \end{cases}$$

$L_1 + L_2$ $-2y - 6z = 0$ $y = -3z$

L_2 : $3x = 3z + 5z$

$3x = 8z$

$x = \frac{8}{3} z$

$$\overset{X}{\begin{pmatrix} x \\ y \\ z \end{pmatrix}} = \begin{pmatrix} \frac{8}{3}z \\ -3z \\ z \end{pmatrix} = \frac{z}{3} \overset{U}{\begin{pmatrix} 8 \\ -9 \\ 3 \end{pmatrix}}$$

$E_2 = \mathrm{Vect}(U)$ $\dim E_2 = 1$

$$E_{-2} = \{ X \in \mathbb{R}^3 \,/\, CX = -2X \}$$

$$(C + 2I_3) X = 0$$

$$\begin{pmatrix} 4 & 0 & 0 \\ -3 & 1 & -1 \\ 3 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 4x = 0 \\ -3x + y - z = 0 \\ 3x + y - z = 0 \end{cases} \qquad \begin{cases} x = 0 \\ y = z \end{cases}$$

$$\overset{X}{\begin{pmatrix} x \\ y \\ z \end{pmatrix}} = \begin{pmatrix} 0 \\ y \\ y \end{pmatrix} = y \overset{V}{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}}$$

$E_{-2} = \text{vect}(V)$ $\dim E_{-2} = 1$

c) C n'est pas diagonalisable car

$\dim E_{-2} = 1 < 2 =$ l'ordre de $\lambda = -2$

par suite $\dim E_2 + \dim E_{-2} = 1 + 1 = 2 < 3 = \dim \mathbb{R}^3$