Chapitre 1 : Variables aléatoires

- Expérience aléatoire
  C'est une expérience où le hasard intervient
- Variable aléatoire
  résultat de l'expérience aléatoire

VA continue ↙ ↘ VA Discrète

I - Variable aléatoire discrète :

1/ Densité de Probabilité (DDP)

l'ensemble des proba de tout les résultats possibles.

Exp 

X	1	2	3	4	5	6
	$P(x=1)$	$P(x=2)$				
$P(X=x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

DDP

1ère condition d'une DDP : $P(X=x) \geq 0$

2 - $\sum_{i=1}^{6} P(X=x_i) = 1$

2/ Fonction de répartition (cumulative)

est le cumul des proba à un certain point $F(x) = P(X \leq a)$

	1	2	3	4	5	6
$P(X \leq x)$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	1

$\lim_{x \to +\infty} F(x) = 1$

$P(X \leq a) = F(a)$

$\lim_{x \to -\infty} F(x) = 0$

Exp $P(X \leq 3) = F(3) = \frac{3}{6}$

$P(a \leq X \leq b) = F(b) - F(a)$
$= P(X \leq b) - P(X \leq a)$

3/ Esperance (moyenne)

$$E(x) = \sum_{i=1}^{6} x_i \, P(X = x_i)$$

$$E(x) = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + \dots + 6 \, \frac{1}{6}$$

Variance.

$$\frac{1}{n} \sum (x_i - \bar{x})^2$$

$$E(x_i - E(x_i))^2 = V(x)$$ ⇐ Stat descri

$$E(x_i^2 - 2x_i E(x_i) + E(X_i)^2$$

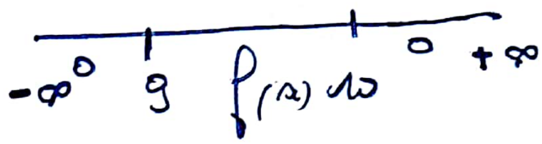
$$E(x_i^2) - \underbrace{2\, E(x_i) E(x_i)}_{E(x_i^{\,2})} + E(x_i)$$

$$V(x) = E(x_i^2) - [E(x_i)]^2$$

DPP

1ère condition

$$f(x) \geqslant 0$$



2ème condition

$$\int_9^{10} f(x)\, dx = 1$$

Fonction de répartition

Si $x < 9 \Rightarrow F(x) = \int_{-\infty}^{x} f(t)\, dt = 0$

Si $9 < x \leqslant 10 \Rightarrow F(x) = \int_{-\infty}^{9} f(t)\, dt + \int_{9}^{x} f(t)\, dt$

Si $10 < x \Rightarrow F(x) = \int_{-\infty}^{9} f(t)\, dt + \int_{9}^{10} f(t)\, dt + \int_{10}^{x} f(t)\, dt = 1$

Espérance

Cas discret $E(X) = \sum x P(X = x)$

Cas continu $E(X) = \int_{-\infty}^{+\infty} x f(x)\, dx$

Variance

$V(X) = E(X^2) - [E(X)]^2$

cas discret $E(X^2) = \sum x^2 P(X = x)$

cas continu $E(X^2) = \int_{-\infty}^{+\infty} x^2 f(x)\, dx$

Exercice 1 :

Soit $f(x) = \begin{cases} Kx(1-x) & \forall\ 0 \leq x \leq 1 \\ 0 & \text{sinon} \end{cases}$

1) Déterminer (K) pour que $f(x)$ soit une DDP
2) Calculer la fonction de répartition
3) Calculer $E(X)$ et $V(X)$

$-\infty$ 0 $Kx(1-x)$ 1 $+\infty$

1ère condition : $f(x) \geq 0 \Rightarrow \underset{\geq 0}{k}\ \underset{\geq 0}{x\ (1-x)} \geq 0 \Rightarrow k \geq 0$

2ème condition $\int_{-\infty}^{+\infty} f(x)\, dx = 1 \Rightarrow \underbrace{\int_{-\infty}^{0} f(x)\, dx}_{0} + \int_0^1 f(x)\, dx + \underbrace{\int_1^{+\infty} f(x)\, dx}_{0} = 1$

$\Rightarrow \int_0^1 k\, x(1-x)\, dx = 1.$

$\Rightarrow K \int_0^1 x(1-x)\, dx \Rightarrow k \int_0^1 x - x^2\, dx = 1.$

$\Rightarrow k \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 1 \Rightarrow k\left[\left(\frac{1^2}{2} - \frac{1^3}{3}\right) - \left(\frac{0^2}{2} - \frac{0^3}{3}\right)\right]$

$\Rightarrow 1 \quad \Rightarrow k\left(\frac{1}{6}\right) = 1 \quad \Rightarrow k = 6$

2) Si $x < 0 \Rightarrow F(x) = \int_{-\infty}^{+\infty} f(t)\, dt = 0$

Si $0 \leq x < 1 \Rightarrow F(x) = \underbrace{\int_{-\infty}^{0} f(t)\, dt}_{0} + \int_0^x f(t)\, dt$

$\int_0^x 6t(1-t)\, dt = 6 \int_0^x t - t^2\, dt$

Rappel Primitives

$\int C^{st}\, dx = [C^{st} x]$

$\int x\, dx = \left[\frac{x^2}{2}\right]$

$\int x^n\, dx = \left[\frac{x^{n+1}}{n+1}\right]$

$\int t\, dt = \left[\frac{t^2}{2}\right]$

$\int x\, dt = [x\, t]$

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$$\Rightarrow 6\left[\frac{t^2}{2}-\frac{t^3}{3}\right]_0^x = 6\left[\frac{x^2}{2}-\frac{x^3}{3}\right]$$
$$= 3x^2 - 2x^3$$

Si $x > 1 \Rightarrow F(x) = \int_{-\infty}^{0} \underbrace{f(t)}_{0}\,dt + \int_0^1 f(t)\,dt + \int_1^x \underbrace{f(t)}_{0}\,dt$

$$= \int_0^1 6t\,(1-t)\,dt = 6\int_0^1 t - t^2\,dt$$
$$= 6\left[\left(\frac{1^2}{2}-\frac{1^3}{3}\right) - \left(\frac{0^2}{2}-\frac{0^3}{3}\right)\right]$$
$$= 6\left(\frac{1}{6}\right) = 1$$

$F(x) = P(X \leqslant x)$

3)

$$E(x) = \int_{-\infty}^{+\infty} x f(x)\,dx = \int_0^1 x\; 6x(1-x)\,dx = 6\int_0^1 x^2 - x^3\,dx$$
$$= 6\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = 6\left(\frac{1}{3}-\frac{1}{4}\right) = \frac{1}{2}$$

$$V(x) = E(x^2) - [E(x)]^2 = \frac{6}{20} - \frac{1}{4} = \frac{1}{20}$$

$$E(x^2) = \int_0^1 x^2 f(x)\,dx = \int_0^1 x^2 . 6x(1-x)\,dx = 6\int_0^1 x^3 - x^4\,dx$$
$$= 6\left[\frac{x^4}{4} - \frac{x^5}{5}\right]_0^1 = 6\left(\frac{1}{4}-\frac{1}{5}\right) = \frac{6}{20}$$

Exercice 2.

Soit $f(x) = \begin{cases} ke^{-\frac{x}{2}} & \forall\ x \geqslant 0 \\ 0 & \text{sinon} \end{cases}$

1) Déterminer (k) pour que $f(x)$ soit une DDP
2) calculer la fonction de répartition $F(x)$
3) Calculer $E(x)$
4) Soit $y = -e^{-\frac{x}{2}}$ Prouvez que y suit une loi uniforme $[0, 1]$

$-\infty \qquad 0 \qquad ke^{-\frac{x}{2}} \qquad +\infty$

1ère condition

$f(x) \geqslant 0 \Rightarrow ke^{-\frac{x}{2}} \geqslant 0 \Rightarrow k \geqslant 0$

2ème condition

$$\int_0^{+\infty} ke^{-\frac{x}{2}}\,dx = 1 \Rightarrow k\int_0^{+\infty} e^{-\frac{x}{2}}\,dx$$

f

$$\Rightarrow k\left[\frac{1}{-\frac{1}{2}}\,e^{-\frac{x}{2}}\right]_0^{+\infty} = 1 \Rightarrow k\left[-2\,(0-1)\right] = 1.$$
$$\Rightarrow 2k = 1 \Rightarrow k = \frac{1}{2}$$

Primitives

$\int e^x\,dx = [e^x]$

$\int e^f.dx = \left[\frac{1}{f'}.e^f\right]$

$\int f'.e^f = [e^f]$

$e^{+\infty} = +\infty$

$e^{-\infty} = 0$

$e^0 = 1$

$e^1 = 1$

Si $x < 0 \Rightarrow F(x) = \int_{-\infty}^{x} f(t)\,dt = 0$

Si $x > 0 \Rightarrow F(x) = \underbrace{\int_{-\infty}^{0} f(t)\,dt}_{0} + \int_{0}^{x} f(t)\,dt$

$= \int_{0}^{x} \frac{1}{2} e^{-\frac{t}{2}}\,dt = -\left[e^{-\frac{t}{2}}\right]_{0}^{x} = -\left[e^{-\frac{x}{2}} - 1\right] = 1 - e^{-\frac{x}{2}}$

Loi	DDP	Caractéristiques	description
$X \sim B(P)$ Bernoulli	$P(X=x) = P^{x}(1-P)^{1-x}$	$E(X) = P$ $V(X) = P(1-P)$	X : 0 echec, 1 succès l'expérience ne se répète pas
$X \sim B(n,P)$ Binomiale n : nombre de repetitions P : Probabilité de succès	$P(X=x) = C_n^x P^x (1-P)^{1-x}$	$E(X) = nP$ $V(X) = nP(1-P)$	X : 0 echec, 1 succès l'expérience se répète (n) fois
$X \sim G(P)$ Geométrique	$P(X=x) = P(1-P)^{x-1}$	$E(X) = \frac{1}{P}$ $V(X) = \frac{1-P}{P^2}$	X : 0 échec, 1 succès on répète l'experience jusqu'à l'obtention pour la Première fois succès (actualiser PB)
$X \sim H(N,n,K)$ H hypergeometrique	$P(X=k) = \frac{C_{N_1}^{k} \; C_{N_2}^{n-k}}{C_N^n}$ $P(X=3) = \frac{C_4^3 C_6^1}{C_{10}^4}$	$E(X) =$ $V(X) =$	N : Population N_1 : ceux qui ont le k N_2 : ceux qui n'ont pas le k n : échantillon k : ceux qui ont le k dans l'échantillon
$X \sim P(\lambda)$ Poisson	$P(X=x) = \frac{e^{-\lambda} \lambda^{x}}{x!}$	$E(X) = \lambda$ $V(X)$ λ	Nbre de client servi dans un axe de temp
$X \sim E(\lambda)$ Exponentielle	$f(x) = \lambda e^{-\lambda x}$	$E(X) = \frac{1}{\lambda}$ $V(X) = \frac{1}{\lambda^2}$	temps pour servir les clients
$X \sim U[a,B]$ uniforme	$f(x) = \frac{1}{b-a}$	$E(X) = \frac{a+b}{2}$ $V(X) = \frac{(b-a)^2}{12}$	
$X \sim N(m, \sigma^2)$ Normale	$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-m)^2}{2\sigma^2}}$	$E(X) = m$ $V(X) = \sigma^2$	

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Variable centrée (loi normale)

X centrée ssi $E(X) = 0$

X non centrée $\rightarrow E(X) = m \neq 0$

Pour centrer $X \rightarrow Z = X - m$

$E(Z) = E(X - m) = E(X) - E(m) = m - m = 0$

$E(a\,X) = a\,E(X)$

$E(b) = b$

$E(aX + b) = a\,E(X) + b$

$V(aX) = a^2\,V(X)$

$V(b) = 0$

Variable réduite

X réduite ssi $V(X) = 1$

X non réduite $V(X) = \sigma^2 \neq 1$

Pour réduire $X \rightarrow Z = \frac{X}{\sqrt{V(X)}} = \frac{X}{\sigma}$

$$V(Z) = V\left(\frac{X}{\sigma}\right) = V\left(\frac{1}{\sigma}X\right) = \frac{1}{\sigma^2}V(X)$$
$$= \frac{1}{\sigma^2}\sigma^2 = 1$$

Variable centrée réduite

$X \sim N(m, \sigma^2)$ $\quad E(X) = m \neq 0$

$V(X) = \sigma^2 \neq 1$

$$Z = \frac{X - m}{\sigma} \sim N(0, 1)$$

Exercice 4.

$Y \sim N(10, \sigma^2)$

$U \sim N(5, 2^2)$

$V \sim N(5, 5^2)$

$P(7 \leq y \leq 13) = 0,3$

1) $\Rightarrow P\left(\frac{7-10}{\sigma} \leq \frac{Y-10}{\sigma} \leq \frac{13-10}{\sigma}\right) = 0,3$

$P\left(\frac{-3}{\sigma} \leq Z \leq \frac{3}{\sigma}\right) = 0,3$

$P(Z \leq 1,24) = F(1,24) = 0,8925$

$F(2,1) = 0,9821$

$F(1) = 0,8413$

$F(-2,35) = 1 - F(2,35)$ $F(-a) = 1 - F(a)$

$F(7,82) = 1$ $P(0 > 3,8) = 1$

$F(a) = 0,975$ $a = 1,96$

$F(b) = 0,1587$ $F(-b) = 1 - 0,1587 = 0,8413$ $\Rightarrow -b = 1$

$b = -1$

$P(-2,13 < x < 2,13) = 2F(2,13) - 1.$

$P(-a < x < a) = F(a) - F(-a)$

$= F(a) - (1 - F(a))$

$= 2F(a) - 1$

$F(a) = F(b)$ ⚠ $N(0,1)$

$a = b$

$- P\left(\frac{-3}{\sigma} \leq Z \leq \frac{3}{\sigma}\right) = 0,3$

$2F\left(\frac{3}{\sigma}\right) - 1 = 0,3$

$F\left(\frac{3}{\sigma}\right) = 0,65$

$\frac{3}{\sigma} = 0,39 \Rightarrow \sigma \frac{3}{0,39} = 7,\dots$

2) $P(U \leq 10) = P(V \leq k) \Rightarrow P\Big(\underbrace{\frac{U-5}{2}}_{Z \sim N(0,1)} \leq \frac{10-5}{2}\Big) = P\Big(\underbrace{\frac{V-5}{5}}_{Z \sim N(0,1)} \leq \frac{k-5}{5}\Big)$

$\Rightarrow P(Z \leq 2,5) = P\left(Z \leq \frac{k-5}{5}\right) \Rightarrow F(2,5) = F\left(\frac{k-5}{5}\right)$

$\rightarrow 2,5 = \frac{k-5}{5} \Rightarrow k = 17,5$

RQ:

Si $X_1 \sim N(m_1, \sigma_1^2)$

$X_2 \sim N(m_2, \sigma_2^2)$

$X_1 + X_2 \sim N(m_1 + m_2, \sigma_1^2 + \sigma_2^2)$

$X_1 - X_2 \sim N(m_1 - m_2, \sigma_1^2 + \sigma_2^2)$, $V(X_1 - X_2) = V(X_1) + V(X_2) - 2\underbrace{Cov(X_1, X_2)}_{0}$

$\alpha X_1 \sim N(\alpha m_1, \alpha^2 \sigma_1^2)$

$V(X_1 + X_2) = V(X_1) + V(X_2) + 2 Cov(X_1 X_2)$

$V(X_1 - X_2) = V(X_1) + V(X_2) - 2 Cov(X_1 X_2)$

Exercice 5:

$X \sim N(m, \sigma^2)$ Tq $P(1 < x < 10)$

$P(x < 3) = 0,1587$

$P(x > 12) = 0,0228$

$P\left(\frac{1-m}{\sigma} < \frac{x-m}{\sigma} < \frac{10-m}{\sigma}\right)$

$P(x < 3) \overset{\text{cette valeur}}{=} 0,1587 \Rightarrow P\left(\underbrace{\frac{x-m}{\sigma}}_{Z \sim N(0,1)} < \frac{3-m}{\sigma}\right) = 0,1587$

$F\left(\frac{3-m}{\sigma}\right) \overset{1}{=} 0,1587$

$F\left(\frac{m-3}{\sigma}\right) = 0,8413$

$\frac{m-3}{\sigma} = 1 \Rightarrow \sigma = m - 3 = 3$

$P(x > 12) = 0,0228$

$P(x < 12) = 1 - 0,0228 = 0,9772$

$P\left(\frac{x-m}{\sigma} < \frac{12-m}{\sigma}\right) = 0,9772 \Rightarrow F\left(\frac{12-m}{\sigma}\right) = 0,9772 \Rightarrow \frac{12-m}{\sigma} = 2$

$12 - m = 2\sigma$

$12 - m = 2(m-3)$

$12 - m = 2m - 6$

$18 = 3m$

$m = 6$

$X \sim N(6, 3^2)$

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$$X \sim N(6, 3^2)$$

$$P\left(\frac{1-m}{\sigma} < \frac{X-m}{\sigma} < \frac{10-m}{\sigma}\right)$$

$$P\left(\frac{1-6}{3} < \frac{x-6}{3} < \frac{10-6}{3}\right)$$

$$P\left(\frac{-5}{3} < Z < \frac{4}{3}\right) = F\left(\frac{4}{3}\right) - F\left(\frac{-5}{3}\right) = F(1,33) - F(-1,66) = F(1,33) - (1 - F(1,66))$$

$$= 0,9082 - 1 + 0,9515$$

Exercice 3

$$f(x)\begin{cases} k e^{-\frac{x}{2}} & \text{si } x \geq 0 \\ 0 & \text{sinon}\end{cases}$$

1/ $k = \frac{1}{2}$

2/ $F(x) = 1 - e^{-\frac{x}{2}}$

$$E(x) = \int_0^{+\infty} x \cdot \frac{1}{2} e^{-\frac{x}{2}} dx \qquad \int f' e^f = e^f.$$

$$-\int -\frac{1}{2} e^{-\frac{x}{2}}$$

$$\begin{cases} U = x \\ V' = \frac{1}{2} e^{-\frac{x}{2}}\end{cases} \begin{cases} U' = 1 \\ V = -e^{-\frac{x}{2}}\end{cases} \qquad = -\left[e^{-\frac{x}{2}}\right]$$

$$[U.V]_0^{+\infty} - \int U.V$$

$$= \left[x\left(-e^{-\frac{x}{2}}\right)\right]_0^{+\infty} - 2\int_0^{+\infty} -\frac{1}{2} e^{-\frac{x}{2}} dx$$

$$= 0 - 2\left[e^{-\frac{x}{2}}\right]_0^{+\infty} = -2(0-1) = 2$$

3/ $X \to f(x) = \frac{1}{2} e^{-\frac{x}{2}}$

$X \sim \mathcal{E}\left(\frac{1}{2}\right) \Rightarrow E(X) = \frac{1}{\lambda} = \frac{1}{\frac{1}{2}} = 2 \qquad \lambda e^{-\lambda x}$ avec $\lambda = \frac{1}{2}$

4/ $y = e^{-\frac{x}{2}} \sim U[0,1]$

$X \sim U[a,b] \to f(x) = \frac{1}{b-a}$

$Y \sim U[0,1] \to g(y) = \frac{1}{1-0} = 1.$

$Y = f(x)$ → fonction de repartition de $G(Y)$

→ DDP de y. (deriva)

→ $E(Y) = \int y \cdot g(y)$

$\sqrt{}$ Log

e

Poly

$$\boxed{F(x) = \int_{-\infty}^{x} f(x) \Rightarrow f(x) = \partial F(x)}$$ densité de P

2/6/7

$F(x) = 1 - e^{-\frac{x}{2}} \quad \forall x > 0$

on pose $G(Y) = P(Y \leq y)$

$= P(e^{-\frac{x}{2}} \leq y)$

$= P(-\frac{x}{2} \leq \ln y)$

$= P(x \geq -2\ln y)$

$= 1 - P(x < -2\ln y)$

$= 1 - F(-2\ln y)$

$= 1 - (1 - e^{-\frac{-2\ln y}{2}}) = y \Rightarrow g(y) = G'(y) = 1 \Rightarrow y \sim U[0,1]$

$x \geq 0$

$\frac{-x}{2} \leq 0$

$0 \leq e^{-\frac{x}{2}} \leq 1$

$0 \leq y \leq 1$

Exercice :

Soit $f(x) = \begin{cases} \frac{k}{x} \log x & \text{si } 1 \leq x \leq e \\ 0 & \text{sinon} \end{cases}$

1/ Déterminer (k) pour que $f(x)$ soit une DDP

2/ calculer la fonction de rep de x

3/ Soit $y = \log$ calculer $E(Y)$

0 | 1 | $\frac{k}{x}\log x$ | e | 0

1ère condition

$f(x) \geq 0 \Rightarrow \frac{k}{x} \ln x \geq 0 \Rightarrow k \geq 0$

2ème condition

$\int_1^e \frac{k}{x} \ln x \, dx = 1 \Rightarrow k \int_1^e \underbrace{\frac{1}{x}}_{U'} \underbrace{\ln x}_{U} \, dx$

$\int U'U = [\frac{1}{2} U^2]$

$\Rightarrow k[\frac{1}{2}(\ln x)^2]_1^e = 1 \Rightarrow k[\frac{1}{2}(\ln e)^2 - \frac{1}{2}(\ln 1)^2]$

$\Rightarrow k \cdot \frac{1}{2} = 1 \Rightarrow \boxed{k = 2}$.

Si $x \leq 1 \Rightarrow F(x) = \int_{-\infty}^{x} f(t)\,dt = 0$

Si $1 < x \leq e \Rightarrow F(x) = \int_{-\infty}^{1} \underbrace{f(t)}_{0}\,dt + \int_{1}^{x} f(t)\,dt$

$= \int_{1}^{x} \frac{2}{t} \operatorname{Ln} t\,dt = 2 \int_{1}^{x} \frac{1}{t} \operatorname{Ln} t\,dt = 2\left[\frac{1}{2}(\operatorname{Ln} t)^2\right]_1^x$

$= (\operatorname{Ln} x)^2 - (\underbrace{\operatorname{Ln} 1}_{0})^2 = (\operatorname{Ln} x)^2$

Si $x > 2 \Rightarrow F(x) = 1$

$y = \operatorname{Ln} x$

$1 < x < e$

$\operatorname{Ln} 1 < \operatorname{Ln} x < \operatorname{Ln} e$

$0 < Y < 1$

on pose $G(y) = P(Y \leq y)$

$= P(\operatorname{Ln} x \leq y) = P(x \leq e^y)$

$= F(e^y) = (\operatorname{Ln} e^y)^2$

$= y^2 \to g(y) = G'(y) = 2y$

$E(y) = \int_0^1 y\, g(y)\,dy$

$= \int_0^1 y \cdot 2y\,dy = 2\left[\frac{y^3}{3}\right]_0^1 = 2 \cdot \frac{1}{3} = \frac{2}{3}$

$= 2 \int_0^1 y^2\,dy$

$$F(x) = \begin{cases} 0 & \text{si } x \leq 1 \\ (\operatorname{Ln} x)^2 & \text{si } 1 \leq x \leq e = 2,7 \\ 1 & \text{si } x \geq e = 2,7 \end{cases}$$

$F(-1) = 0$

$F(0) = 0$

$F(3) = 1$

$F(4) = 1$

$F(2) = \operatorname{Ln} 2$

Exercice 1 :

1/ $F(x) = 3x^2 - 2x^3$

$y = -|x - \frac{1}{2}|$

$0 \leq x < 1$

$-\frac{1}{2} < x - \frac{1}{2} < \frac{1}{2}$

$0 < |x - \frac{1}{2}| < \frac{1}{2}$

$0 > -|x - \frac{1}{2}| > -\frac{1}{2}$

$0 > y > -\frac{1}{2}$

$|x| < a \Rightarrow -a < x < a$

on pose $G(y) = P(Y \leq y)$

$= P(-|x - \frac{1}{2}| \leq y)$

$= P(|x - \frac{1}{2}| \geq -y) = 1 - P(|x - \frac{1}{2}| \leq -y) = 1 - P(y \leq x - \frac{1}{2} < -y)$

$= 1 - P(y + \frac{1}{2} \leq x \leq \frac{1}{2} - y)$

$= 1 - (F(\frac{1}{2} - y) - F(y + \frac{1}{2}))$

2/

$G(y) = 1 - \left[\left(3\left(\frac{1}{2} - y\right)^2 - 2\left(\frac{1}{2} - y\right)^3\right) - \left(3\left(y + \frac{1}{2}\right)^2 - 2\left(y + \frac{1}{2}\right)^3\right)\right]$

Exercice 2:

1) $X \sim \mathcal{E}(\lambda)$ $\begin{cases} f(x) = \lambda e^{-\lambda x} \\ E(X) = \frac{1}{\lambda} = 5 \Rightarrow \lambda = \frac{1}{5} = 0,2 \\ V(X) = \frac{1}{\lambda^2} \end{cases}$

2) $P(X<2) = F(2) = 1 - e^{-\frac{2}{5}}$

$F(x) = 1 - e^{-\lambda x}$
$= 1 - e^{-\frac{x}{5}}$

$P(X>6) = 1 - P(X<6) = 1 - F(6)$
$= 1 - 1 + e^{-6/5}$
$= e^{-6/5}$

$P(3<X<8) = F(8) - F(3)$
$= 1 - e^{-8/5} - 1 + e^{-\frac{3}{5}}$
$= e^{\frac{-3}{5}} - e^{-\frac{8}{5}}$

$\lambda e^{-\lambda x}$

si $x<0 \Rightarrow F(x) = \int_{-\infty}^{x} f(t)\,dt = 0$

si $x>0 \Rightarrow F(x) = \int_{-\infty}^{0} \underbrace{f(t)\,dt}_{0} + \int_{0}^{x} f(t)\,dt$
$= \int_{0}^{x} \lambda e^{-\lambda t}\,dt = -\left[e^{-\lambda t}\right]_0^x = -(e^{-\lambda x} - 1)$
$= 1 - e^{-\lambda x}$

3) $P(X<5) = 0,5$
$F(5) = 1 - e^{-\frac{5}{5}} = 1 - e^{-1} \neq 0,5$.

Exercice 6 :

1/ X : la demande $\sim N(200, 30^2)$

$P(X>230) = 1 - P(X<230)$
$= 1 - P\left(\frac{X-200}{30} < \frac{230-200}{30}\right)$
$= 1 - P(Z<1) = 1 - F(1) = 1 - 0,8413$
$= 0,1587$

2/ de temps

$P(X > \text{stock}) < 0,05$

$P(X > \text{stock}) = 0,05$

$P(X < \text{stock}) = 0,95$

$P\left(\frac{X-200}{300} < \frac{\text{stock} - 200}{30}\right) = 0,95$

$$F\left(\frac{stock - 200}{30}\right) = 0,95$$

$$\frac{stock - 200}{30} = 1,65$$

$$stock \simeq 249$$

Exercice 7 :

X : la durée de vie

$X \sim N(7, 2^2)$

1) $P(X < 2) = P\left(\frac{X-7}{2} < \frac{2-7}{2}\right)$

$= P\left(Z < -\frac{5}{2}\right) = F(-2,5) = 1 - F(2,5)$

$= 1 - 0,9938$

$= 0,0062.$

nbre $= 100\,000 \times 0,0062$

$= 620$

2/ $P(X<4) = P\left(\frac{X-7}{2} < \frac{4-7}{2}\right)$

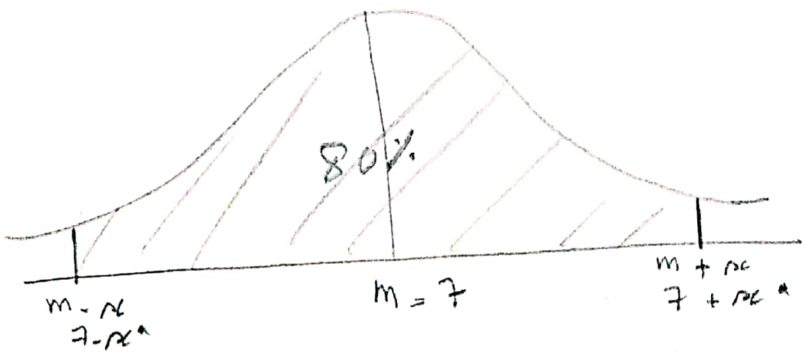
$= P\left(Z < -\frac{3}{2}\right) = F\left(-\frac{1}{5}\right) = 1 - F(1,5)$

$= 1 - 0,9332$

$= 0,0668$

$coef = \frac{0,0668}{0,0068} = 10, \dots$

3)



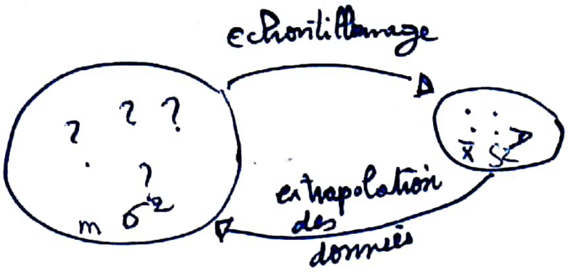
$P(7 - x^* < X < 7 + x^*) = 0,8$

$P\left(\frac{7 - x^* - 7}{2} < \frac{X - 7}{2} < \frac{7 + x^* - 7}{2}\right) = 0,8$

$P\left(\frac{-x^*}{2} < Z < \frac{x^*}{2}\right) = 0,8$

$\Rightarrow 2F\left(\frac{x^*}{2}\right) - 1 = 0,8 \Rightarrow F\left(\frac{x^*}{2}\right) = 0,9 \Rightarrow \frac{x^*}{2} = 1,29 \Rightarrow x^* = 2,58$

Echantillonnage



$E_1\ (x_1 x_2 \dots x_n) < \bar{x} = \frac{1}{n}\sum x_i \quad S^2 = \frac{1}{n}\sum (x_i - \bar{x})^2$

$E_2\ (x_1\ x_2\ \dots\ x_n) < \bar{x}\ S^2$

$\vdots$

$E_p\ (x_1 x_2 \dots\ x_n) < \bar{x}^2\ S^2$

Soit $(x_1 x_2 \dots x_n)$ un echantillon iid (independament et identiquement distribué)

La loi $\chi^2_{(n)}$ Khi 2

$x_i \sim N(0, 1)$

$x_i^2 \sim \chi^2(1)$

$\sum_{i=1}^{n} x_i^2 \sim \chi^2_{(n)} \Rightarrow \sum_{i=1}^{n} N(0, 1)^2 - \chi^2_{(n)}$

$\bar{x}$

La loi $\chi^2_{(n)}$ Khi 2

Soit un echantillon $(x_1 x_2 \dots x_n)$ iid

$x_i \sim N(0, 1)$

$x_i^2 \sim \chi^2_{(1)}$ degré de liberté

$\sum_{i=1} N(0,1)^2 \sim \chi^2(n)$

$E(\chi^2_{(n)}) = n$

$V(\chi^2(n)) = 2n$

Exp:

Soit $x_i \sim N(m, \sigma^2)$ avec $n = 25$

$y = \frac{1}{\sigma^2}\sum (x_i - m)^2$

1) Determiner la loi de y calculer $E(y)$ et $V(y)$

2) a - $P(y \leq a) = 0{,}1 \rightarrow 1 - P(y \geq a) = 0{,}1$ $P(y \geq a) = 0{,}9$ $a = 16{,}47$.

b - $P(y \geq b) = 0{,}95$

$: - P(y \geq 24{,}34) = c$

$x_i \sim N(m, \sigma^2)$

$\frac{x_i - m}{\sigma} \sim N(0,1)$

$\left(\frac{x_i - m}{\sigma}\right)^2 \sim \chi^2_{(1)}$

$\sum_{i=1}^{n} \left(\frac{x_i - m}{\sigma}\right)^2 \sim \chi^2_{(n)}$

$y = \frac{1}{\sigma^2} \sum (x_i - m)^2 \sim \chi^2_{(n)}$

$E(y) = n = 25$

$V(y) = 2n = 50$

Exercice 5:

$x_1 \ldots x_n \sim N(2,4)$ avec $n = 25$

1) $\bar{x} = \frac{1}{n}\sum x_i \sim N\left(2, \frac{4}{25}\right)$

2) $P(\bar{x} < 6) = P\left(\frac{\bar{x}-2}{\sqrt{\frac{4}{25}}} < \frac{6-2}{\sqrt{\frac{4}{25}}}\right) = P(Z < 10) = F(10) = 1.$

$P(\bar{x} < 12) = P\left(\frac{\bar{x}-2}{\sqrt{\frac{4}{25}}} < \frac{12-2}{\sqrt{\frac{4}{25}}}\right) = P(Z < 25) = F(25) = 1.$

$P(\bar{x} > 1) = 1 - P(\bar{x} < 1) = 1 - P\left(\frac{\bar{x}-2}{\frac{2}{5}} < \frac{1-2}{\frac{2}{5}}\right) = 1 - P(Z < -2{,}5) = 1 - F(-2{,}5)$
$= 1 - (1 - F(2{,}5))$
$= F(2{,}5) = 0{,}9938$

f

a $y = \frac{1}{k}\sum (x_i - 2)^2$

Par identification $\frac{1}{\sigma^2}\sum (x_i - m)^2 \sim \chi^2_{(n)}$

$k = \sigma^2 = 4$

$E(y) = 25 = n \qquad V(y) = 2n = 50$

c) $P(y \geq a) = 0{,}95 \Rightarrow a = 14{,}61.$

b) $P(y \leq b) = 0{,}1 \Rightarrow b = 16{,}47.$

$P\left(\frac{1}{\sigma}\sum (x_i - \bar{x})^2 \geq \frac{c}{\sigma^2}\right) = 0{,}9$

$P\left(\chi^2_{(n-1)} \geq \frac{c}{\sigma^2}\right) = 0{,}9$

$\frac{c}{\sigma^2} = 15{,}66$

$C = 15{,}66 \times 4$

3/ compléter remarque

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Suite Ex 5 :

$X_i \sim N(2,4)$ $n = 25$ $X \sim N(m, \sigma^2)$ $\bar{X} \sim N\left(m, \frac{\sigma^2}{n}\right)$

σ^2 inconnu et m connu $\sum x_i = 50$ $\sum x_i^2 = 316$

$$P(\bar{X} \leq 3) = P\left(\frac{\bar{x} - m}{\sqrt{\frac{\sigma^2}{n}}} < \frac{3 - m}{\sqrt{\frac{\sigma^2}{n}}}\right) = P\left(\frac{\bar{x} - m}{\underbrace{\sqrt{\frac{S^2}{n}}}_{S_{(n)}}} < \frac{3-2}{\sqrt{\frac{S^2}{25}}}\right) = P\left(S_{(25)} < \frac{1}{\sqrt{\frac{8,64}{25}}}\right)$$

$$= P\left(S_{(25)} < 1,7\right) = 0,1$$

avec $S^2 = \frac{1}{n} \sum (x_i - m)^2 = \frac{1}{n} \sum^n (x_i^2 - 2x_i m + m^2) = \frac{1}{n} \left(\sum x_i^2 - 2m \sum x_i + n m^2\right) = \frac{1}{25}\left(316 - 2,2,50 + 25,2^2\right)$

$= \frac{216}{25} = 8,64$.

=

x??

$x_i \sim N(0,1)$

$x^2 \sim \chi^2_{(1)}$

$\sum x_i^2 \sim \chi^2_{(n)}$

$\sum_i^n N(0,1)^2 \sim \chi^2_{(n)}$

$E(\chi^2_{(n)}) = n$

$V(\chi^2_{(n)}) = 2n$

①

la loi student

$x_i \sim N(0,1)$ $\quad y_i \sim \chi^2_{(n)}$

$E(T) = 0 \quad V(T) = \frac{n}{n-2}$

$$T = \frac{N(0,1)}{\sqrt{\frac{\cancel{\chi^2}_{(n)}}{n}}} = \frac{x_i}{\sqrt{\frac{y_i}{n}}} \sim S_{(n)}$$

②

$$\frac{\bar{x} - m}{\sqrt{\frac{\sigma^2}{n}}} \to N(0,1) \qquad \frac{\bar{x} - m}{\sqrt{\frac{S^2}{n}}} \to S(1) \qquad \frac{\bar{x} - m}{\sqrt{\frac{S'^2}{n}}} \Rightarrow S(n-1)$$

σ^2 connu

σ^2 inconnu, m connu

σ^2 inconnu, m inconnu

$x_i \sim N(m, \sigma^2)$ $\begin{cases} E(x_i) = m \\ V(x_i) = \sigma^2 \end{cases}$

$$\bar{x} = \frac{1}{n} \sum x_i \sim N\left(m, \frac{\sigma^2}{n}\right) \begin{cases} E(\bar{x}) = m \\ V(\bar{x}) = \frac{\sigma^2}{n} \end{cases}$$

④

$$S^2 = \frac{1}{n} \sum (x_i - m)^2$$

m connu

$$E(S^2) = \frac{(n-1)\sigma^2}{n}$$

$$V(S^2) = \frac{2\sigma^4}{n}$$

$$Z = \frac{n S^2}{\sigma^2} \sim \chi^2_{(n)}$$

⑤

$$S'^2 = \frac{1}{n-1} \sum (x_i - \bar{x})^2$$

m inconnu

$$E(S'^2) = \sigma^2$$

$$V(S'^2) = \frac{2\sigma^4}{n-1}$$

$$Z = \frac{(n-1)S'^2}{\sigma^2} \sim \chi^2_{(n-1)}$$

$x_i \sim N(m, \sigma^2) \to \bar{x} \sim N\left(m, \frac{\sigma^2}{n}\right)$

$$Z = \frac{\bar{x} - m}{\sqrt{\frac{\sigma^2}{n}}} = \frac{\bar{x} - m}{\frac{\sigma}{\sqrt{n}}} = \sqrt{n}\,\frac{\bar{x} - m}{\sigma} \sim N(0,1)$$

⑦

$x_i \sim N(0,1) \to \bar{x} \sim N\left(0, \frac{1}{n}\right)$

$$Z = \frac{\bar{x} - 0}{\sqrt{\frac{1}{n}}} = \frac{\bar{x}}{\frac{1}{\sqrt{n}}} = \sqrt{n}\,\bar{x} \sim N(0,1)$$

Khi 2

⑧

$x_i \sim N(m, \sigma^2)$

Khi 2

$$\frac{1}{\sigma^2} \sum (x_i - m)^2 \sim \chi^2_{(n)}$$

$$\frac{1}{\sigma^2} \sum (x_i - \bar{x})^2 \sim \chi^2_{(n-1)}$$

Case 2 : Exemple.

$x_i \sim N(m, \sigma^2)$

$$T = \frac{x_i - m}{\sqrt{\frac{1}{n}\sum(x_i - m)^2}}$$

Déterminer la loi de T

Donner E(T) V(T)

calculer

a) $P(T \geq a) = 0,99 \Rightarrow a = 2,7874$

b) $P(T \leq b) = 0,1 \longrightarrow b = 1,7081$

n = 25 donnée

$$T = \frac{\overbrace{\frac{x_i - m}{\sigma}}^{\rightarrow N(0,1)}}{\sqrt{\frac{\frac{1}{\sigma^2}\sum(x_i - m)^2}{n}}} = \sqrt{\frac{\chi^2(n)}{n}} \sim S_{(n)}$$

$E(T) = 0 \quad V(T) = \frac{n}{n-2} = \frac{25}{23}$

a) Case 4 :

Caractéristiques de $\bar{x}$

$x_i \sim N(\underset{E(x)}{m}, \underset{V(x)}{\sigma^2})$

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \sim N\left(\underset{E(\bar{x})}{m}, \underset{V(\bar{x})}{\frac{\sigma^2}{n}}\right)$$

$$E(\bar{x}) = E\left(\frac{1}{n}\sum x_i\right) = \frac{1}{n}\sum_{i=1}^{n} E(x_i) = \frac{1}{n}\boxed{\sum_{i=1}^{n}} m = \frac{1}{n}\boxed{n}\, m = m$$

$$E(a\,x) = a\,E(x) \;/\; E(\sum x_i) = \sum E(x_i) \;/\; \sum_{i=1}^{n} C^{st} = n\,C^{st}$$

$$V(\bar{x}) = V\left(\frac{1}{n}\sum x_i\right) = \frac{1}{n^2}\sum_{i=1}^{n} V(x_i) = \frac{1}{n^2}\sum_{i=1}^{n}\sigma^2 = \frac{1}{n^2}\, n\sigma^2 = \frac{\sigma^2}{n}$$

$$V(a\,x) = a^2\,V(x) \;/\; V(b) = 0 \;/\; V(\sum x_i) = \sum V(x_i)$$

σ^2 : variance de l'échantillon

$$\sigma^2 \leftarrow S^2 = \frac{1}{n}\sum(x_i - \bar{x})^2$$

$$S^2 = \frac{1}{n}\sum(x_i - m)^2 \qquad S'^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

m connu — m inconnu.

$$E(S^2) = \frac{(n-1)\sigma^2}{n} \qquad E(S'^2) = \sigma^2$$

$$V(S^2) = \frac{2\sigma^4}{n} \qquad V(S'^2) = \frac{2\sigma^4}{n-1}$$

* $\frac{1}{\sigma^2}\sum(x_i - \bar{x})^2$. Si $X_i \sim N(0,1)$

$$\frac{1}{\sigma^2}\sum(x_i^2 - 2x_i\bar{x} + \bar{x}^2)$$

$\sum x_i$

$\sum a x_i = a\sum x_i$

$\sum \bar{x} = n\bar{x}$

$$\frac{1}{\sigma^2}\left(\sum_{i=1}^{n} x_i^2 - 2\bar{x}\overbrace{\sum_{i=1}^{n} x_i}^{n\bar{x}} + n\bar{x}^2\right)$$

$$\bar{x} = \frac{\sum x_i}{n} \Rightarrow \sum x_i = n\bar{x}$$

$$\frac{1}{\sigma^2}\Big(\underbrace{\sum^n x_i^2}_{\chi^2(n)} - \underbrace{n\bar{x}^2}_{\chi^2(1)}\Big) \sim \chi^2(n-1)$$

Case 5, 6

$$\frac{nS^2}{\sigma^2} = \frac{n\,1}{\sigma^2\, n}\sum(x_i - m)^2$$

$$Z = \frac{nS^2}{\sigma^2} = \frac{1}{\sigma^2}\sum(x_i - m)^2 \sim \chi^2(n)$$

$$\frac{(n-1)S'^2}{\sigma^2} = \frac{1\,(n-1)}{(n-1)\,\sigma^2}\sum(x_i - \bar{x})^2$$

$$Z = \frac{(n-1)S'^2}{\sigma^2} = \frac{1}{\sigma^2}\sum(x_i - \bar{x})^2 \sim \chi^2(n-1)$$

Case 3 : formules du student.

* $\dfrac{\bar{x} - m}{\sqrt{\frac{\sigma^2}{n}}} \sim N(0,1)$

* $$\frac{\bar{x} - m}{\sqrt{\frac{S^2}{n}}} = \frac{\bar{x} - m}{\sqrt{\frac{\frac{1}{n}\sum(x_i - m)^2}{n}}} = \frac{\bar{x} - m}{\frac{1}{\sqrt{n}}\sqrt{\frac{\sum(x_i - m)^2}{n}}} = \frac{\sqrt{n}\,\overbrace{\frac{\bar{x} - m}{\sigma}}^{\to N(0,1)}}{\underbrace{\sqrt{\frac{\frac{1}{\sigma^2}\sum(x_i - m)^2}{n}}}_{\to \sqrt{\frac{\chi^2(n)}{n}}}} \sim S(n)$$

* $$\frac{\bar{x} - m}{\sqrt{\frac{S'^2}{n}}} = \frac{\bar{x} - m}{\sqrt{\frac{\frac{1}{n-1}\sum(x_i - \bar{x})^2}{n}}} = \frac{\sqrt{n}\,\overbrace{\frac{\bar{x} - m}{\sigma}}^{\to N(0,1)}}{\underbrace{\sqrt{\frac{\frac{1}{\sigma^2}\sum(x_i - \bar{x})^2}{n-1}}}_{\to \sqrt{\frac{\chi^2(n-1)}{n-1}}}} \sim S(n-1)$$

Chapitre : Estimateur

$E_1 (x_1 x_2 \ldots x_n) \left\langle \begin{matrix} \bar{x} \\ S^2 \end{matrix} \right) \Rightarrow \hat{\theta}_1$

$E_2 (x_1 x_2 \ldots x_n) \left\langle \begin{matrix} \bar{x} \\ S^2 \end{matrix} \right) \Rightarrow \hat{\theta}_2$

$E_p (x_1 x_2 \ldots x_n) \left\langle \begin{matrix} \bar{x} \\ S^2 \end{matrix} \right) \Rightarrow \hat{\theta}_3$

Sans biais

espérance

$E(\hat{\theta}) = \theta$

$E(\hat{\theta}) \neq \theta \Rightarrow \hat{\theta}$ biaisé

$\lim_{n \to +\infty} E(\hat{\theta}) = \theta$

$\Rightarrow \hat{\theta}$ est asymptotiquement sans biais

Convergent

$\lim_{n \to +\infty} V(\theta) = 0$

RQ : Si $\hat{\theta}_1$ et $\hat{\theta}_2$ deux estimateurs sans biais et convergent $V(\hat{\theta}_1) \leq V(\hat{\theta}_2)$

$\hat{\theta}_1$ est meilleur que $\hat{\theta}_2$

Efficace Rao et Cramer

$V(\hat{\theta}) = \frac{1}{I_n(\hat{\theta})} \leftarrow$ quantité d'information de fisher

$I_n(\hat{\theta}) = -E\left(\frac{\partial^2 \operatorname{Log} L(x_1, x_2 \ldots x_n)}{\partial \theta^2} \right)$

avec $L(x_1, \ldots x_n, \theta)$ est la fonction de vraisemblance

$L(x_1, x_2, \ldots x_n, \theta) = P(X = x_1) \cdot P(X = x_2) P(X = x_3) \ldots P(X = x_n)$

$= \prod_{i=1} P(X = x_i) = \prod f(x_i, \theta)$

Exp

Soit $X_i \sim P(\lambda) = \begin{cases} P(X = x) = \frac{e^{-\lambda} \lambda^{x_i}}{x_i!} \\ E(X) = \lambda \\ V(X) = \lambda \end{cases}$

Soit $\hat{\lambda} = \bar{x} = \frac{1}{n} \sum^{n} x_i$ Prouvez que $\bar{x}$ est efficace.

sans biais

$E(\hat{\lambda}) = \lambda$??

$E(\hat{\lambda}) = E(\bar{x}) = E\left(\frac{1}{n} \sum^{n} x_i\right) = \frac{1}{n} \sum^{n} E(x_i) = \frac{1}{n} \sum^{n} \lambda = \frac{1}{n} n \cdot \lambda = \lambda$

$E(ax) = a E(x) / E(\sum x_i) = \sum E(X) / \sum^{n}$ cst $= n$ cst

Convergence $\lim_{n \to +\infty} V(\hat{\lambda}) = 0$

$V(\hat{\lambda}) = V(\bar{x}) = V\left(\frac{1}{n} \sum x_i\right) = \frac{1}{n^2} \sum^n \lambda = \frac{1}{n^2} n \cdot \lambda = \frac{\lambda}{n}$ Variance ? $\prod = \sum$?

$V(ax) = a^2 V(x) \mid V(\sum x_i) = \sum V(x_i)$

$\lim_{n \to +\infty} V(\hat{\lambda}) = \lim_{n \to +\infty} \frac{\lambda}{n} = 0 \Rightarrow \hat{\lambda}$ convergent.

Efficace

$V(\hat{\lambda}) = \frac{1}{I_n(\hat{\lambda})}$ avec $I_n(\hat{\lambda}) = -E\left(\frac{\partial^2 \log L(x_1 \dots x_n)}{\partial \lambda^2}\right)$

1) $L(x_1, \dots x_n, \lambda) = \prod_{i=1}^{n} \frac{e^{-\lambda} \lambda^{x_i}}{x_i !}$

2) $\log L(x_1 \dots x_n, \lambda) = \log\left(\prod_{i=1} \frac{e^{-\lambda} \lambda^{x_i}}{x_i !}\right)$

$= \sum \log\left(\frac{e^{-\lambda} \lambda^{x_i}}{x_i !}\right)$

$= \sum^n (-\lambda + x_i \log \lambda - \log x_i !)$

$= -n\lambda + \sum^n x_i \log \lambda - \sum^n \log x_i !)$

3)

$\frac{\partial \log L(x_1 \dots x_n, \lambda)}{\partial \lambda} = -n + \frac{\sum x_i}{\lambda}$

4) $\frac{\partial^2 \log L(x_1 \dots x_n, \lambda)}{\partial \lambda^2} = -\frac{\sum x_i}{\lambda^2}$

5)

$I_n(\hat{\lambda}) = -E\left(\frac{\partial^2 \log L(x_1 \dots x_n, \lambda)}{\partial \lambda^2}\right)$

$= -E\left(-\frac{\sum x_i}{\lambda^2}\right) = \frac{\sum^n E(x_i)}{\lambda^2}$

$= \frac{\sum^n \lambda}{\lambda^2} = \frac{n\lambda}{\lambda^2} = \frac{n}{\lambda}$

or $V(\hat{\lambda}) = \frac{\lambda}{n} = \frac{1}{\frac{n}{\lambda}} = \frac{1}{I_n(\lambda)} \Rightarrow$ l'efficace.

$\log a.b = \log a + \log b$

$\log \prod = \sum \log$

$\log \frac{a}{b} = \log a - \log b$

$\log a^n = n \log a$

$\log \frac{1}{a} = -\log a$

$\log e^x = x$

$\partial \log x = \frac{1}{x}$

$\partial \log U = \frac{U'}{U}$

Estimation Ponctuelle

Estimateur de maximum de vraisemblance EMVS

$$\begin{cases} \dfrac{\partial \log L(x_1, x_2 \dots x_n, \theta)}{\partial \theta} = 0 \\ \dfrac{\partial^2 \log L(x_1 \dots x_n, \theta)}{\partial \theta^2} < 0 \end{cases}$$

$$\partial \frac{1}{f} = -\frac{f'}{f^2}$$

$$\partial \frac{1}{x} = -\frac{1}{x^2}$$

Exp $X \sim P(\lambda)$

$$\frac{\partial \log L(x_1 \dots x_n, \lambda)}{\partial \lambda} = -n + \frac{\sum x_i}{\lambda} = 0$$

$$= \frac{\sum x_i}{\lambda} = n$$

$$\frac{\partial^2 \log L(x_1 \dots x_n, \lambda)}{\partial \lambda} = \frac{-\sum x_i}{\lambda} < 0 \qquad \lambda = \frac{\sum x_i}{n} = \bar{X}$$

Exercice :

Soit $X \sim B(P) = \begin{cases} P(X = x) = p^x (1-p)^{1-x} \\ E(X) = P \\ V(X) = P(1-P) \end{cases}$ bernoulli

Determiner EMVS $\hat{P}$ de P

est-il efficace.

$$\begin{cases} \dfrac{\partial \log L(x_1 \dots x_n, P)}{\partial P} = 0 \\ \dfrac{\partial^2 \log L(x_1 \dots x_n, P)}{\partial P} < 0 \end{cases}$$

$$L(x_1 \dots x_n, P) = \prod_{i=1}^{n} p^{x_i} (1-p)^{1-x_i}$$

$$\log L(x_1 \dots x_n, P) = \log \prod_{i=1}^{n} p^{x_i} (1-p)^{1-x_i}$$

$$= \sum_{i=1}^{n} \log \left(p^{x_i} (1-p)^{1-x_i}\right)$$

$$= \sum^{n} \left(x_i \log P + (1 - x_i) \log (1-P)\right)$$

$$\sum_{i=1}^{n} x_i \log P + \sum_{i=1}^{n} (1 - x_i) \log (1-P)$$

$$\frac{\partial \log L(x_i \dots x_n, P)}{\partial P} = \frac{\sum x_i}{P} - \frac{n - \sum x_i}{(1-P)} = 0$$

$$\frac{\sum x_i (1-P) - P(1 - \sum x_i)}{P(1-P)} = 0$$

$$\sum x_i - P\cancel{\sum x_i} - nP + P\cancel{\sum x_i} = 0$$

$$\sum x_i = nP \Rightarrow \hat{P} = \frac{\sum x_i}{n} = \bar{x}$$

$$\frac{\partial^2 \log L(x_1, \dots x_n, P)}{\partial P} = -\frac{\sum x_i}{P^2} - \frac{(n - \sum x_i)}{(1-P)^2}$$

$X \sim B(P)$

$\hat{P} = \bar{x} = \frac{\sum x_i}{n} < 1 \qquad n > \sum x_i \qquad \text{prob} < 1$

Sans biais

$E(\hat{P}) = P$

$E(\hat{P}) = E(\bar{x}) = E\left(\frac{1}{n}\sum^{n} x_i\right) = \frac{1}{n}\sum^{n} E(x_i)$

$= \frac{1}{n}\sum^{n} P = \frac{1}{n} n P = P.$

convergent

$\lim_{n \to +\infty} V(\hat{P}) = 0$

$V(\hat{P}) = V(\bar{x}) = V\left(\frac{1}{n}\sum x_i\right) = \frac{1}{n^2}\sum^{n} V(x_i)$

$= \frac{1}{n^2}\sum^{n} P(1-P) = \frac{nP(1-P)}{n^2} = \frac{P(1-P)}{n}$

$\lim \frac{P(1-P)}{n} = 0.$

$I_n(\hat{P}) = -E\left(-\frac{\sum x_i}{P^2} - \frac{(n-\sum x_i)}{(1-P)^2}\right)$

$= \frac{\sum E(x)}{P^2} + \frac{(n - \sum E(x))}{(1-P)^2}$

$= \frac{nP}{P^2} + \frac{n-nP}{(1-P)^2} = \frac{n}{P(1-P)}$

or $V(\hat{P}) = \frac{P(1-P)}{n} = \frac{1}{\frac{n}{P(1-P)}} = \frac{1}{I_n(\hat{P})}$

$\Rightarrow \hat{P}$ efficace.

$x \sim \mathcal{E}\left(\frac{1}{\theta}\right) = \begin{cases} f(x) = \frac{1}{\theta} e^{-\frac{x}{\theta}} \\ E(x) = \frac{1}{\frac{1}{\theta}} = \theta \\ V(x) = \frac{1}{\frac{1}{\theta^2}} = \theta^2 \end{cases}$

$x \sim \mathcal{E}(\lambda) \begin{cases} f(x) = \lambda e^{-\lambda x} \\ E(x) = \frac{1}{\lambda} \\ V(x) = \frac{1}{\lambda^2} \end{cases}$

1) déterminer $\hat{\theta}$ de θ MMVS (maximum de vraisemblance)

2) Est-il efficace.

Réponse :

1- $L(x_1, x_2 \ldots x_n, \theta) = \prod_{i=1}^{n} \frac{1}{\theta} e^{-\frac{x}{\theta}}$

$\log L(x_1, x_2 \ldots x_n, \theta) = \log \prod_{i=1}^{n} \frac{1}{\theta} e^{-\frac{x}{\theta}}$

$= \sum \log\left(\frac{1}{\theta} e^{-\frac{x_i}{\theta}}\right)$

$= \sum\left(-\log\theta - \frac{x_i}{\theta}\right)$

$= -n\log\theta - \frac{\sum x_i}{\theta}$

(3) $\frac{\partial \log L(x_1 \ldots x_n, \theta)}{\partial\theta} = \frac{-n}{\theta} + \frac{\sum x_i}{\theta^2} = 0$

$\frac{-n\theta + \sum x_i}{\theta^2} = 0 \Rightarrow \sum x_i = n\theta \Rightarrow \hat{\theta} = \frac{\sum x_i}{n} = \bar{x}$

$\partial^2 \log L(x_1 \ldots x_n, \theta) = \frac{n}{\theta^2} - \frac{2\sum x_i}{\theta^3} = \frac{n\theta - 2\sum x_i}{\theta^3}$

$$\frac{-n\frac{\sum x_i}{n} - 2\sum x_i}{\theta^3} = -\frac{\sum x_i}{\theta^3} < 0$$

Sans biais

$E(\hat\theta) = \theta$.

$E(\hat\theta) = E(\bar x) = E\left(\frac{1}{n}\sum x_i\right) = \frac{1}{n}\sum E(x_i) = \frac{1}{n}\cdot\sum^n \theta = \frac{1}{n}\cdot n\theta = \theta$ Sans biais

convergence

$\lim\limits_{n\to+\infty} V(\hat\theta) = 0$

$V(\hat\theta) = V\left(\frac{1}{n}\sum x_i\right) = \frac{1}{n^2}\sum^n V(x) = \frac{1}{n^2}\cdot\sum^n \theta^2$

$= \frac{1}{n^2} n\cdot\theta^2 = \frac{\theta^2}{n} \Rightarrow \lim\limits_{x\to+\infty}\frac{\theta^2}{n} = 0$

Efficacité:

$I_n(\hat\theta) = -E\left(-\frac{\sum x_i}{\theta^2}\right) = \frac{\sum E(x_i)}{\theta^3}$

$= \frac{\sum^n \theta}{\theta^3} = \frac{n\theta}{\theta^3} = \frac{n}{\theta^2}$

$a\,v(\hat\theta) = \frac{\theta^2}{n}$.

Exercice 8 série contrôle 2014

$X \sim N(4, \theta)$ $\begin{cases} f(x) = \frac{1}{\sqrt{2\pi\theta}} e^{-\frac{(x-m)^2}{2\theta}} \\ E(x) = 4 = m \\ V(x) = \theta \end{cases}$ $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-m)^2}{2\sigma^2}}$

$L(x_1 x_2 \dots x_n, \theta) = \prod\limits_{i=1}^n \frac{1}{\sqrt{2\pi\theta}} e^{-\frac{(x-m)^2}{2\theta}}$

$\mathrm{Lg}\,L(x_1 x_2 \dots x_n, \theta) = \sum\limits_{i=1}^n \mathrm{Log}\left(\frac{1}{\sqrt{2\pi\theta}} e^{-\frac{(x_i-m)^2}{2\theta}}\right)$

$= \sum^n\left(-\frac{1}{2}\log 2\pi\theta - \frac{(x_i-m)^2}{2\theta}\right)$

$= -\frac{n}{2}\log 2\pi\theta - \frac{\sum(x_i-m)^2}{2\theta}$

$\frac{\partial \log L(x_1 \dots x_n, \theta)}{\partial\theta}$

$= -\frac{n}{2}\frac{2\pi}{2\pi\theta} + \frac{\sum(x_i-m)^2}{2\theta^2}$

$= \frac{-n\theta + \sum(x_i-m)^2}{2\theta^2} = 0$

$\sum(x_i-m)^2 = n\theta$

$\boxed{\hat\theta = \frac{1}{n}\sum(x_i-m)^2 = S^2}$

Explication
$\begin{cases} \frac{\partial\log L(x_1 \dots x_n, \theta)}{\partial\theta} = 0 \\ \frac{\partial^2 \log L(x_1 \dots x_n\theta)}{\partial\theta^2} < 0 \end{cases}$ ~~$-\frac{n\theta}{2\theta^2}$~~

$\frac{\partial^2\log L(x_1 \dots x_n, \theta)}{\partial\theta^2} = \frac{n}{2\theta^2} - \frac{2\sum(x_i-m)^2}{2\theta^3}$

$= \frac{n\theta - 2\sum(x_i-m)^2}{2\theta^3} = \frac{n\frac{1}{n}\sum(x_i-m)^2 - 2\sum(x_i-m)^2}{2\theta^3} = \frac{-\sum(x_i-m)^2}{2\theta^3} < 0$

Sans biais

$E(\hat{\theta}) = E(S^2) = \frac{(n-1)}{n}\sigma^2 \neq \sigma^2 \Rightarrow$ biaisé

$\lim_{n \to +\infty} E(\hat{\theta}) = \lim_{n \to +\infty} (1 - \frac{1}{n})\sigma^2 = \lim_{x \to +\infty} \sigma^2 - \underbrace{\frac{\sigma^2}{n}}_{\to 0} = \sigma^2$

$\hat{\theta}$ est asymptotiquement sans biais

Convergence:

$\lim \overset{\text{variance}}{V}(\hat{\theta}) = \lim V(S^2) = \lim_{n \to +\infty} \frac{2\sigma^4}{n} = 0$

$2^{ème}$ méthode

Sans biais

$E(\hat{\theta}) = E(S^2) = E\left(\frac{1}{n}\sum(x_i - m)^2\right)$

$= \frac{1}{n}\sigma^2 E\left(\frac{1}{\sigma^2}\sum(x_i - m)^2\right)$

$= \frac{1}{n}\sigma^2 E(\chi^2(n))$

$= \frac{1}{n}\sigma^2 n = \sigma^2 \Rightarrow S^2$ sans biais

Convergence

$\lim_{n \to +\infty} V(\hat{\theta}) = 0$

$V(\hat{\theta}) = V\left(\frac{1}{n}\sum(x_i - m)^2\right)$

$= \frac{1}{n^2}\sigma^4 V\left(\frac{1}{\sigma^2}\sum(x_i - m)^2\right)$

$= \frac{1}{n^2}\sigma^4 V(\chi^2(n))$

$= \frac{1}{n^2}\sigma^4 \cdot 2n = \frac{2\sigma^4}{n}$

$\lim \frac{2\sigma^2}{n} = 0$

le 2e test

$E(\hat{\theta}) \neq \theta \Rightarrow$ le biais $= E(\hat{\theta}) - \theta$

$\lim_{n \to +\infty} E(\hat{\theta}) = \theta$

Exercice (Sp 2015)

$x_1, x_2 \dots x_n$ un échantillon iid admettant une fonction de répartition

$$F(x) = \begin{cases} 1 - x^{-\frac{1}{\theta}} & x > 1 \quad \theta < \frac{1}{2} \\ 0 & \text{sinon} \end{cases}$$

Gpas une DDP

1) Calculer $E(x)$ et $V(x)$ (3)

2) Déterminer $\hat\theta$. EMVS de θ (2)

3) on pose $y = \log x$. Donner la loi de y (3)

En déduire $E(y)$ et $V(y)$

4) Déterminer les propriétés de $\hat\theta$ (S.B, convergent, efficace) (4)

• $$f(x) = \begin{cases} \frac{1}{\theta} x^{-\frac{1}{\theta}-1} & x \geq 1 \\ 0 & \text{sinon} \end{cases}$$

$\partial x^n = n x^{n-1}$

$\int x^n = \frac{x^{n+1}}{n+1}$

$$E(x) = \int_1^{+\infty} x \cdot \frac{1}{\theta} x^{-\frac{1}{\theta}-1} \, dx = \frac{1}{\theta} \int_1^{+\infty} x^{-\frac{1}{\theta}} \, dx = \frac{1}{\theta} \cdot \left[\frac{x^{-\frac{1}{\theta}+1}}{-\frac{1}{\theta}+1} \right]_1^{+\infty}$$

$$= \frac{1}{\theta} \left[\frac{1}{x^{\frac{1}{\theta}-1}} \cdot \frac{1}{(-\frac{1}{\theta}+1)} \right]_1^{+\infty}$$

$$= \frac{1}{\theta} \left(0 - \frac{1}{-\frac{1}{\theta}+1} \right)$$

$$= \frac{1}{\theta} \frac{1}{\frac{1}{\theta}-1} = \frac{1}{\theta} \frac{1}{\frac{1-\theta}{\theta}} = \boxed{\frac{1}{1-\theta}}$$

$$V(x) = E(x^2) - [E(x)]^2$$

$$= \frac{1}{1-2\theta} - \left[\frac{1}{1-\theta} \right]^2$$

$$E(x^2) = \int_1^{+\infty} x^2 \frac{1}{\theta} x^{-\frac{1}{\theta}-1} \, dx$$

$$= \frac{1}{\theta} \int_1^{+\infty} x^{-\frac{1}{\theta}+1} \, dx = \frac{1}{\theta} \left[\frac{x^{-\frac{1}{\theta}+2}}{-\frac{1}{\theta}+2} \right]_1^{+\infty} = \frac{1}{\theta} \left[\frac{1}{x^{\frac{1}{\theta}-2}} \frac{1}{-\frac{1}{\theta}+2} \right]_1^{+\infty} = \frac{1}{\theta} \left[0 - \frac{1}{-\frac{1}{\theta}+2} \right]$$

$$= \frac{1}{\cancel{\theta}} \cdot \left(\frac{1}{\frac{1-2\theta}{\cancel{\theta}}} \right) = \frac{1}{1-2\theta}.$$

2) $$L(x_1, \dots x_n, \theta) = \prod_{i=1}^{n} \frac{1}{\theta} x^{-\frac{1}{\theta}-1}$$

$$\log L(x_1 \dots x_n, \theta) = \sum \log \left(\frac{1}{\theta} x_i^{-\frac{1}{\theta}-1} \right)$$

$$= \sum^n \left(-\log\theta + \left(-\frac{1}{\theta} - 1\right) \log x_i \right)$$

$$= -n \log\theta + \left(-\frac{1}{\theta} - 1 \right) \sum_{i=1}^{n} \log x_i$$

$$= -n \log\theta - \frac{\sum \log x_i}{\theta} - \sum^n \log x_i$$

$$\frac{\partial \log(x_1 \ldots x_n,\theta)}{\partial\theta} = -\frac{n}{\theta} + \frac{\sum \log x_i}{\theta^2} = 0$$

$$\frac{-n\theta + \sum \log x_i}{\theta^2} = 0$$

$$\sum \log x_i = n\theta \Rightarrow \boxed{\hat\theta = \frac{\sum \log x_i}{n}}$$

$$\frac{\partial^2 \log L(x_1 \ldots x_n,\theta)}{\partial\theta} = \frac{n}{\theta^2} - \frac{2\sum \log x_i}{\theta^3}$$

$$= \frac{n\theta - 2\sum \log x_i}{\theta^3} = \frac{n\frac{1}{n}\sum \log x_i - 2\sum \log x_i}{\theta^3}$$

$$= \frac{-\sum \log x_i}{\theta^3} < 0$$

3/ on pose $G(y) = P(Y \leq y)$

$$= P(\log x \leq y)$$

$$= P(x \leq e^y) = P(e^y)$$

$$\boxed{= 1 - e^{-\frac{y}{\theta}}}$$

$\partial x^n = n x^{n-1}$

$\lambda e^{-\lambda x}$

Exponentielle $E(x) = \frac{1}{\lambda}$ $V(x) = \lambda$

$$g(y) = G'(y) = \frac{1}{\theta} e^{-\frac{y}{\theta}}$$

$y \sim \mathcal{E}\left(\frac{1}{\theta}\right)$ $E(y) = \frac{1}{\frac{1}{\theta}} = \theta$

$$V(y) = \frac{1}{\frac{1}{\theta^2}} = \theta^2$$

Sans biais

$$E(\hat\theta) = E\left(\frac{\sum \log x_i}{n}\right) = \frac{1}{n}\sum^n E(\log x_i)$$

$$= \frac{1}{n}\sum^n E(y) = \frac{1}{n}\sum^n \theta = \frac{1}{n} n \cdot \theta$$

$\Rightarrow \hat\theta$ sans biais

Convergence

$$V(\hat\theta) = V\left(\frac{1}{n}\sum \log x_i\right) = \frac{1}{n^2}\sum^n V(y)$$

$$= \frac{1}{n^2}\sum^n \theta^2 = \frac{1}{n^2} n\theta^2 = \frac{\theta^2}{n}$$

$\lim \frac{\theta^2}{n} = 0 \Rightarrow \hat\theta$ convergeant.

Efficace

$$I_n(\hat\theta) = -E\left(-\frac{\sum \log x_i}{\theta^3}\right) = \frac{\sum E(y)}{\theta^3} = \frac{n\theta}{\theta^3} = \frac{n}{\theta^2} = \frac{1}{\frac{\theta^2}{n}} = \frac{1}{V(\hat\theta)}$$

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Exercice 6 :

① Poisson
Ex 10 bernoulli (vote)

$X_i \sim N(m, \sigma^2)$

$U = \frac{X_i - m}{\sigma} \sim N(0,1)$

$V = \sqrt{n}\,\frac{\bar{X} - m}{\sigma} \sim N(0,1)$

$Z = \frac{\sum (x_i - \bar{x})^2}{\sigma^2} \sim \chi^2_{(n-1)}$

$T = \sqrt{n}\,\frac{\bar{X} - m}{S'} = \frac{\bar{X} - m}{\sqrt{\frac{S'^2}{n}}} \sim S_{(n-1)}$

$X \sim N(m, \sigma^2) \to \bar{X} \sim N\left(m, \frac{\sigma^2}{n}\right)$

$\frac{\bar{X} - m}{\sqrt{\frac{\sigma^2}{n}}} = \frac{\bar{X} - m}{\frac{\sigma}{\sqrt{n}}} = \sqrt{n}\,\frac{\bar{x} - m}{\sigma} \sim N(0,1)$

$n\,\frac{(\bar{x} - m)^2}{\sigma^2} \sim \chi^2_{(1)}$

2/ $(X_1 \cdots X_n) \sim N(m_1, \sigma_1^2)$

$(Y_1 \cdots Y_n) \sim N(m_2, \sigma_2^2)$

$\bar{X} \sim N\left(m_1, \frac{\sigma_1^2}{n_1}\right)$ $\quad \bar{x} - \bar{y}$

$\bar{y} \sim N\left(m_2, \frac{\sigma_2^2}{n_2}\right)$

$\bar{x} - \bar{y} \sim N\left(m_1 - m_2, \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}\right)$

$\bar{x} - \bar{y} \sim N\left(0, \frac{1}{2}\right)$

$P(\bar{x} - \bar{y} \leq 6) = P\left(\frac{\bar{x} - \bar{y} - (m_1 - m_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \leq \frac{6 - 0}{\sqrt{\frac{1}{2}}}\right)$

$= P(Z < 6\sqrt{2}) = F(6\sqrt{2}) = 1$.

$\sigma_1^2 = \sigma_2^2 = \sigma^2$ inconnu

$\bar{X} \sim N\left(m_1, \frac{\sigma^2}{n_1}\right)$ $\quad \bar{y} \sim N\left(m_2, \frac{\sigma^2}{n_2}\right)$

$\bar{X} - \bar{y} \sim N\left(m_1 - m_2, \sigma^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)\right)$

$P(\bar{x} - \bar{y} \leq 3) = P\left(\underbrace{\frac{\bar{x} - \bar{y} - (m_1 - m_2)}{\sqrt{\sigma^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}}_{S(n_1 + n_2 - 2)} < \frac{3 - (m_1 - m_2)}{\sqrt{\sigma^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}\right)$

2eme

$\boxed{S^* = \frac{(n_1 - 1) S_1'^2 + (n_2 - 1) S_2'^2}{n_1 + n_2 - 2}} = \frac{9{,}95 + 9.100}{18} = 97{,}5$

$P\left(S_{(18)} \leq \frac{3 - 0}{\sqrt{\frac{97{,}5}{5}}}\right) = P(S_{(18)} < 0{,}68) = 0{,}4$.

(Table)

II. Estimation par intervalle de confiance

1) $IC(m) \setminus \sigma^2$ connu $x_i \sim N(m, \sigma^2)$

1ère étape:

$$\hat{m} = \bar{x} \sim N\left(m, \frac{\sigma^2}{n}\right)$$

2ème étape:

$$Z = \frac{\bar{x} - m}{\sqrt{\frac{\sigma^2}{n}}} \sim N(0,1)$$

3ème étape

on pose z_1 et $z_2 \setminus P(z_1 < Z < z_2) = 1 - \alpha$ Niveau de confiance, risque d'erreur

or $N(0,1)$ symetrique P/R à 0 (par rapport)

$$\Rightarrow P(-z < Z < z) = 1 - \alpha \Rightarrow 2F(z) - 1 = 1 - \alpha.$$

$$\Rightarrow F(z) = 1 - \frac{\alpha}{2} \Rightarrow z = T^{able}_{N(0,1)}$$

Encadrement

$$-z < Z < z$$

$$-z < \frac{\bar{x} - m}{\sqrt{\frac{\sigma^2}{n}}} < z.$$

$$-z\sqrt{\frac{\sigma^2}{n}} < \bar{x} - m < z\sqrt{\frac{\sigma^2}{n}}$$

$$-\bar{x} - z\sqrt{\frac{\sigma^2}{n}} < -m < -\bar{x} + z\sqrt{\frac{\sigma^2}{n}}$$

$$\bar{x} - z\sqrt{\frac{\sigma^2}{n}} < m < \bar{x} + z\sqrt{\frac{\sigma^2}{n}}$$

2) $IC(m) / \sigma^2_m$ connu $x_i \sim N(m, \sigma^2)$

1ère étape

$$\hat{m} = \bar{x} \sim N\left(m, \frac{\sigma^2}{n}\right)$$

2ème étape

$$Z = \frac{\bar{x} - m}{\sqrt{\frac{S'^2}{n}}} \sim S_{(n-1)}$$

3ème étape on pose z_1 et $z_2 \setminus P(z_1 < Z < z_2) = 1 - \alpha$ niveau de confiance, risque d'erreur.

or Student symétrique P/R à 0

$$\Rightarrow P(-z < Z < z) = 1 - \alpha \Rightarrow 2F(z) = 1 - \alpha.$$

$$\Rightarrow F(z) = 1 - \frac{\alpha}{2} \Rightarrow z = T^{able}\ \text{student}\ [(n-1), \alpha]$$

Encadrement:

$$-z < Z < z$$

$$-z < \frac{\bar{x} - m}{\sqrt{\frac{S'^2}{n}}} < z.$$

$$-z\sqrt{\frac{S'^2}{n}} < \bar{x} - m < z\sqrt{\frac{S'^2}{n}}$$

$$-\bar{x} - z\sqrt{\frac{S'^2}{n}} < -m < -\bar{x} + z\sqrt{\frac{S'^2}{n}}$$

$$\bar{x} - z\sqrt{\frac{S'^2}{n}} < m < \bar{x} + z\sqrt{\frac{S'^2}{n}}$$

3) IC (σ^2) / m connue

1ère étape

$S^2 = \frac{1}{n} \sum (x_i - m)^2$ ne suit aucune loi

2ème étape

$Z = \frac{nS^2}{\sigma^2} = \frac{1}{\sigma^2} \cdot \sum (x_i - m)^2 \sim \chi^2_{(n)}$

3ème étape

on pose z_1 et $z_2 \setminus P(z_1 < Z < z_2) = 1 - \alpha$.

la $\chi^2_{(n)}$ n'est pas symétrique

$\begin{cases} P(Z > z_1) = 1 - \frac{\alpha}{2} \\ P(Z < z_2) = 1 - \frac{\alpha}{2} \end{cases}$ $\begin{cases} P(Z < z_1) = \frac{\alpha}{2} \Rightarrow z_1 = T\chi^2 \cdot \left[(n), \frac{\alpha}{2}\right] \\ P(Z < z_2) = 1 - \frac{\alpha}{2} \Rightarrow z_2 T\chi^2 \cdot \left[(n), 1 - \frac{\alpha}{2}\right] \end{cases}$

Par la lecture de Table.

4ème étape

$z_1 < Z < z_2$ z_2 toujours $> z_1$

$z_1 < \frac{nS^2}{\sigma^2} < z_2$

$\frac{z_1}{nS^2} < \frac{1}{\sigma^2} < \frac{z_2}{nS^2}$

$\frac{nS^2}{z_2} < \sigma^2 < \frac{nS^2}{z_1}$

4) IC (σ) / m inconnue

1ère étape

$S'^2 = \frac{1}{n-1} \sum (x_i - \bar{x})^2$.

2ème étape

$Z = \frac{(n-1)S'^2}{\sigma^2} = \frac{1}{\sigma^2} \sum (x_i - \bar{x})^2 \sim \chi^2(n-1)$

3ème étape

on pose z_1 et $z_2 \setminus P(z_1 < Z < z_2) = 1 - \alpha$.

la $\chi^2(n-1)$ n'est pas symétrique

$\begin{cases} P(Z > z_1) = 1 - \frac{\alpha}{2} \\ P(Z < z_2) = 1 - \frac{\alpha}{2} \end{cases}$ $\begin{cases} P(Z < z_1) = \frac{\alpha}{2} \Rightarrow z_1 = T\chi^2 \left[(n-1), \frac{\alpha}{2}\right] \\ P(Z < z_2) = 1 - \frac{\alpha}{2} \Rightarrow z_2 T\chi^2 \cdot \left[(n-1), 1 - \frac{\alpha}{2}\right] \end{cases}$

4ème étape

$z_1 < Z < z_2$

$z_1 < \frac{(n-1)S'^2}{\sigma^2} < z_2$

$\frac{z_1}{(n-1)S'^2} < \frac{1}{\sigma^2} < \frac{z_2}{(n-1)S'^2}$

$\frac{(n-1)S'^2}{z_2} < \sigma^2 < \frac{(n-1)S'^2}{z_1}$

Exercice 11 :

① $\sigma^2 = 4$ $\quad X \sim N(m, \sigma^2)$ $\quad n = 25$

1ère étape

$$\hat{m} = \bar{x} \sim N\left(m, \frac{\sigma^2}{n}\right)$$

2ème étape

$$Z = \frac{\bar{x} - m}{\sqrt{\frac{\sigma^2}{n}}} \sim N(0,1)$$

3ème étape

on pose z_1 et z_2 \ $P(z_1 < Z < z_2) = 1 - \alpha$.

or $N(0, 1)$ est symétrique

$$P(-z < Z < z) = 1 - \alpha \Rightarrow F(z) = 1 - \frac{\alpha}{2} = 0{,}975$$

$$z = 1{,}96$$

4ème étape

$$-z < Z < z.$$

$$\bar{x} - z\sqrt{\frac{\sigma^2}{n}} < m < \bar{x} + z\sqrt{\frac{\sigma^2}{n}}$$

$$2 - 1{,}96\sqrt{\frac{4}{25}} < m < 2 + 1{,}96\sqrt{\frac{4}{25}}$$

② a.

1ère étape

$$\hat{m} = \bar{x} \sim N\left(m, \frac{\sigma^2}{n}\right)$$

2ème étape

$$Z = \frac{\bar{x} - m}{\sqrt{\frac{S'^2}{n}}} \sim S_{(n-1)}$$

3ème étape

on pose z_1 et z_2.

$P(z_1 < Z < z_2) = 1 - \alpha$ or le student est symetrique

$P(-z < Z < z) = 1 - \alpha \Rightarrow F(z) = 1 - \frac{\alpha}{2} = 0{,}975 \Rightarrow z = 2{,}064$.

4ème étape

$$\bar{x} - z\sqrt{\frac{S'^2}{n}} < m < \bar{x} + z\sqrt{\frac{S'^2}{n}}$$

$$2 - 2{,}064 \quad < m < 2 + 2{,}064\sqrt{\frac{4{,}16}{n}}$$

$$S'^2 = \frac{1}{n-1}\sum(x_i - \bar{x})^2$$

$$= \frac{1}{n-1}\sum(x_i^2 - 2x_i\bar{x} + \bar{x}^2)$$

$$= \frac{1}{n-1}\left(\sum x_i^2 - 2\bar{x}\sum x_i + n\bar{x}^2\right)$$

$$= \frac{1}{24}\left(200 - 2 \cdot 2 \cdot 50 + 25 \cdot 2^2\right)$$

$$= \frac{100}{24} = 4{,}16.$$

b) I de confiance m inconnue

1ère étape

$S'^2 = \frac{1}{n-1} \sum (x_i - \bar{x})^2$

2ème étape

$Z = \frac{(n-1)S'^2}{\sigma^2} \sim \chi^2_{(n-1)}$

3ème étape

$P(z_1 < Z < z_2) = 1 - \alpha.$

$\begin{cases} P(Z > z_1) = 1 - \frac{\alpha}{2} \\ P(Z < z_2) = 1 - \frac{\alpha}{2} \end{cases} \begin{cases} P(Z < z_1) = \frac{\alpha}{2} = 0{,}05 \Rightarrow z_1 = 13{,}85 \\ P(Z < z_2) = 1 - \frac{\alpha}{2} = 0{,}95 \Rightarrow z_2 = 36{,}42. \end{cases}$

4ème étape

$\frac{(n-1)S'^2}{z_2} < \sigma^2 < \frac{(n-1)S'^2}{z_1}$

$\frac{24 \cdot 4{,}16}{36{,}46} < \sigma^2 < \frac{24 \cdot 4{,}16}{13{,}85}$

Intervalle de confiance de $m_1 - m_2 \mid \sigma_1^2$ et σ_2^2 connues.

$x_1\, x_2 \dots x_{n_1} \sim N(m_1, \sigma_1^2) \rightarrow \bar{x} \sim N(m_1, \frac{\sigma_1^2}{n_1})$

$y_1 y_2 \dots y_{n_2} \sim N(m_2, \sigma_2^2) \rightarrow \bar{y} \sim N(m_2, \frac{\sigma_2^2}{n_2})$

1ère étape

$m_1 - m_2 = \bar{x} - \bar{y} \sim N(\underbrace{m_1 - m_2}_{\text{moyenne}}; \underbrace{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}_{\text{variance}})$

2ème étape

$Z = \frac{(\bar{x} - \bar{y}) - (m_1 - m_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \sim N(0,1)$

3ème étape

on pose z_1 et $z_2 \mid P(z_1 < Z < z_2) = 1 - \alpha$. or $N(0,1)$ est symétrique.

$P(-z < Z < z) = 1 - \alpha \Rightarrow F(z) = 1 - \frac{\alpha}{2}$

$z = T_{N(0,1)}.$

4ème étape

$-z < \frac{(\bar{x} - \bar{y}) - (m_1 - m_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} < z$

$(\bar{x} - \bar{y}) - z\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} < m_1 - m_2 < (\bar{x} - \bar{y}) + z\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$

Intervalle de confiance de $m_1 - m_2 \mid \sigma_1^2 = \sigma_2^2 = \sigma^2$

$x_1 x_2 \dots x_{n_1} \sim N(m_1, \sigma^2) \rightarrow \bar{x} \sim N(m_1, \frac{\sigma^2}{n_1})$

$y_1 \dots y_{n_2} \sim N(m_2, \sigma^2) \rightarrow \bar{y} \sim N(m_2, \frac{\sigma^2}{n_2}).$

1ère étape

$m_1 - m_2 = \bar{x} - \bar{y} \sim N(m_1 - m_2, \sigma^2(\frac{1}{n_1} + \frac{1}{n_2}))$

2ème étape

$Z = \frac{(\bar{x} - \bar{y}) - (m_1 - m_2)}{\sqrt{S'^2(\frac{1}{n_1} + \frac{1}{n_2})}} \sim S_{(n_1 + n_2 - 2)}$

3ème Étape

on pose z_1 et z_2 \ $P(z_1 < Z < z_2) = 1-\alpha$.

or student symetrique

$P(-z < Z < z) = 1-\alpha \Rightarrow F(z) = 1-\frac{\alpha}{2}$

z = Table student (n_1+n_2-2, α)

4ème étape

$$(\bar{x}-\bar{y}) - z\sqrt{S^{*}\left(\frac{1}{n_1}+\frac{1}{n_2}\right)} < m_1 - m_2 < (\bar{x}-\bar{y}) + z\sqrt{S^{*}\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}$$

$$S^{*2} = \frac{(n_1-1)S_1'^2 + (n_2-1)S_2'^2}{n_1+n_2-2}$$

Théorème central limite TCL

$X \sim iid\,(E(x), V(x)) \xrightarrow[n>30]{} X \sim N\,(E(x), V(x))$.

$X \sim P(\lambda) \xrightarrow[n>30]{} X \sim N\,(\lambda, \lambda)$

$X \sim B(P) \xrightarrow[n>30]{} X \sim N(P, P(1-P))$.

$X \sim \chi^2_{(n)} \xrightarrow[n>30]{} X \sim N\,(n, 2n)$

$n = 2500 <$ 1450 votes A ; 1050 votes B

$X \sim B(P) \xrightarrow[n=2500>30]{} X \sim N\,(P, P(1-P))$

1ère étape

$\hat{P} = \bar{x} \sim N\left(P, \frac{P(1-P)}{n}\right)$

2ème étape

$$Z = \frac{\hat{P}-P}{\sqrt{\frac{P(1-P)}{n}}} \sim N(0,1)$$

3ème étape

on pose z_1 et z_2

$F(z) = 1-\frac{\alpha}{2} \Rightarrow z = T_{N(0,1)}$

4ème étape

$-z < Z < z$

$$\hat{P} - z\sqrt{\frac{P(1-P)}{n}} < P < \hat{P} + z\sqrt{\frac{P(1-P)}{n}}$$

III - methode des moments

$$E(x^k) = \frac{1}{n}\sum x_i^k$$

moment théorique d'ordre k = moment empirique d'ordre k.

$X \sim B(P)$. $E(x) = \frac{1}{n}\sum x_i$

$\boxed{\hat{P} = \bar{X}}$

$X \sim P(\lambda)$ $E(x) = \frac{1}{n}\sum x_i$ $\boxed{\hat{\lambda} = \bar{x}}$

$X \sim \mathcal{E}(\lambda) \Rightarrow E(x) = \frac{1}{n}\sum x_i$. $\frac{1}{\lambda} = \bar{x}$ $\boxed{\hat{\lambda} = \frac{1}{\bar{x}}}$

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$X \sim B(n, p)$

$E(X) = \frac{1}{n} \sum x_i$

$np = \bar{x}$

$$\boxed{\hat{p} = \frac{\bar{x}}{n}}$$

$X \sim N(m, \sigma^2)$

$E(X) = \frac{1}{n} \sum x_i$

$E(X^2) = \frac{1}{n} \sum x_i^2$

$V(X) + [E(X)]^2 = \frac{1}{n} \sum x_i^2$

$V(X) = \frac{1}{n} \sum x_i^2 - m^2$

$\sigma^2 = \frac{1}{n} (\sum x_i^2 - n m^2) = S^2$

$V(X) = E(X)^2 - [E(X)]^2$

$S^2 = \frac{1}{n} \sum (x_i - \bar{x})^2$

$= \frac{1}{n} \sum (x_i^2 - 2x_i\bar{x} + \bar{x}^2)$

$= \frac{1}{n} (\sum x_i^2 - 2\bar{x} \underbrace{\sum x_i}_{n\bar{x}} + n x^2)$

$= \frac{1}{n} (\sum x_i^2 - 2n\bar{x}^2 + n\bar{x}^2)$

$= \frac{1}{n} (\sum x_i^2 - n x^2)$

$= \frac{1}{n} (\sum x^2 - n m^2)$

$\bar{x} = \frac{\sum x_i}{n}$

$\sum x_i = n\bar{x}$

Ex :

$X \sim U[2, 3\theta]$

Déterminer θ par méthode de moments.

Est-il convergent

$E(X) = \frac{1}{n} \sum x_i$

$\frac{2 + 3\theta}{2} = \bar{x}$

$1 + \frac{3}{2}\theta = \bar{x}$

$\hat{\theta} = \frac{2}{3} (\bar{x} - 1) = \frac{2}{3}\bar{x} - \frac{2}{3}$

Sans biais

$E(\hat{\theta}) = \frac{2}{3} E(\bar{x}) - \frac{2}{3}$

$= \frac{2}{3} \frac{1}{\cancel{n}} \cancel{n} (1 + \frac{3}{2}\theta) - \frac{2}{3}$

$= \cancel{\frac{2}{3}} + \theta - \cancel{\frac{2}{3}} = \theta \Rightarrow \hat{\theta}$ sans biais !

Ex n2 4 premiers

$(X_1 X_2 \cdots X_{n_1}) \sim N(m_1, \sigma^2)$ $n_1 = 25$ $\sum x_i = 50$ $\sum x_i^2 = 220$.

$(Y_1 Y_2 \cdots Y_{n_2}) \sim N(m_2, \sigma^2)$ $n_2 = 15$ $\sum y_i = 25$ $\sum y_i^2 = 180$

1) $\hat{m}_1 - \hat{m}_2 = \bar{x} - \bar{y}$

$E(\hat{m}_1 - \hat{m}_2) = E(\bar{x} - \bar{y}) = E\left(\frac{1}{n_1}\sum^{n_1} x_i - \frac{1}{n_2}\sum^{n_2} y_i\right) = \frac{1}{n_1}\sum^{n_1} E(x_i) - \frac{1}{n_2}\sum^{n_2} E(y_i)$

$= \frac{1}{n_1}\sum^{n_1} m_1 - \frac{1}{n_2}\sum^{n_2} m_2 = \frac{1}{n_1} n_1 m_1 - \frac{1}{n_2} n_2 m_2 = m_1 - m_2 \Rightarrow \hat{m}_1 - \hat{m}_2$. Sans biais

2) $\sigma'^2 = S^{*2} = \frac{(n_1-1)S_1'^2 + (n_2-1)S_2'^2}{n_1+n_2-2}$.

$E(S^{*2}) = \frac{(n_1-1)E(S_1'^2) + (n_2-1)E(S_2'^2)}{n_1+n_2-2} = \frac{(n_1-1)\sigma^2 + (n_2-1)\sigma^2}{n_1+n_2-2} = \frac{\sigma^2(n_1+n_2-2)}{n_1+n_2-2} = \sigma^2$

Exercice 13 (d'examen)

$X_i \sim N(m, \sigma^2)$ $n = 25$ $\bar{x} = 27{,}5$ $\sum(x_i - \bar{x})^2 = 231$.

1ère étape

$\hat{m} = \bar{x} \sim N\left(m, \frac{\sigma^2}{n}\right)$.

2ème étape

$Z = \frac{\bar{x} - m}{\sqrt{\frac{S'^2}{n}}} \sim S_{(n-1)}$

3ème étape on pose z_1 et z_2 \ $P(z_1 < Z < z_2) = 1-\alpha$. or loi student st symetrique.

$P(-z < Z < z) = 1-\alpha \Rightarrow F(z) = 1 - \frac{\alpha}{2} = 0{,}95$

$S'^2 = \frac{1}{n-1}\sum(x_i - \bar{x})^2 = \frac{231}{24}$ $z = 1{,}7109$.

4ème étape

$-z < Z < z$.

$-z < \frac{\bar{x} - m}{\sqrt{\frac{S'^2}{n}}} < z$.

$\bar{x} - z\sqrt{\frac{S'^2}{n}} < m < \bar{x} + z\sqrt{\frac{S'^2}{n}}$

$27{,}5 - 1{,}711\sqrt{\frac{9{,}625}{25}} < m < 27{,}5 + 1{,}711\sqrt{\frac{9{,}625}{25}}$

[25 ... 28] Sure de 95% que les valeurs possibles sont entre 25 et 28

$\hat{m} = 26{,}5$.

m inconn = S'^2

2) $\begin{cases} H_0 : m = 25 \\ H_1 : m = 30 > 25 \end{cases}$

1ère étape

region de rejet $RR = \{(x_1 x_2 \cdots x_n) \setminus \bar{x} > c\}$

2ème étape: $\alpha = P(RH_0 \setminus H_0 \text{ vrai})$

$0{,}5 = \alpha = P(\bar{x} > c \setminus m = 25)$

$0{,}95 = P(\bar{x} < c \setminus m = 25)$.

$0{,}95 = P\left(\frac{\bar{x} - m}{\sqrt{\frac{S'^2}{n}}} < \frac{c - m}{\sqrt{\frac{S'^2}{n}}} \setminus m = 25\right)$

$= P\left(S_{(n-1)} < \frac{c - 25}{\sqrt{\frac{9{,}615}{25}}}\right) = 0{,}95$

$\frac{c - 25}{\sqrt{\frac{9{,}625}{25}}} =$

Test d'hypothèse

toujours loi normale.

$\begin{cases} H_0 : \text{hypothèse de base} = \text{Pièce déf} \leq 5\% \quad Cp \\ H_1 : \text{hypothèse alternative} = \text{Pièce déf} > 5\% \quad .AP. \end{cases}$

Décision

Réalité	H_0	H_1
H_0	B. Décision	α = erreur
H_1	β = Erreur	B. Décision

α = Erreur de 1ère espace

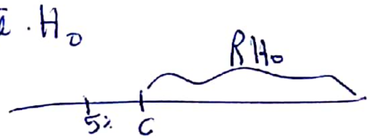
$\alpha = P(RH_0 \setminus H_0 \text{ vrai})$

c = seuil

β = Erreur de 2ème espace.

$\beta = P(RH_1 \setminus H_1 \text{ vrai})$

- Seuil de rejet : C'est le seuil à partir duquel on rejette H_0
- Région de rejet : c'est la région de rejet de H_0.



Exp

tester (m) \ σ^2 connu

$\cdot x_i \sim N(m, \sigma^2) \to \bar{x} \sim N(m, \frac{\sigma^2}{n})$.

$\begin{cases} H_0 : m = 2 \\ H_1 : m = 3 > 2 \end{cases}$

$\cdot \sum x_i = 50$

$n = 25$

$\sigma^2 = 4$

$\alpha = 5\%$

1ère étape : Région de rejet

$R.R = \{(x_1 . x_2 \dots x_n) \setminus \bar{x} > c\}$

2ème étape : seuil de rejet

$\alpha = P(RH_0 \setminus H_0 \text{ vrai})$

$0{,}05 = P(\bar{x} > c \setminus m = 2)$

$0{,}95 = P(\bar{x} < c \mid m = 2)$

$0{,}95 = P\left(\frac{\bar{x} - m}{\sqrt{\frac{\sigma^2}{n}}} < \frac{c - m}{\sqrt{\frac{\sigma^2}{m}}} \middle| m = 2\right)$

$0{,}95 = P\left(z < \frac{c - 2}{\sqrt{\frac{4}{25}}}\right)$.

$\frac{c - 2}{\frac{12}{5}} = 1{,}65 \Rightarrow c = 5{,}96$

3ème étape. Règle de décision

$\bar{x} = \frac{\sum x_i}{n} = \frac{50}{25} = 2 < c = 5{,}96$ Acceptation de H_0

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tester σ^2 / m connue

$$\begin{cases} H_0 : \sigma^2 = 4 \\ H_1 : \sigma^2 = 9 > 4 \end{cases} \qquad n = 25 \qquad \alpha = 0{,}05 \qquad \sum (x_i - m)^2 = 100$$

$$RR = \{(x_1 \dots x_n) \mid S^2 > c\}.$$

2ème étape.

$$\alpha = P(RH_0 \mid H_0 \text{ vrai})$$
$$= P(S^2 > c \mid \sigma^2 = 4)$$
$$= P\left(\frac{nS^2}{\sigma^2} > \frac{nc}{\sigma^2} \mid \sigma^2 = 4\right)$$
$$= P\left(\chi^2(n) > \frac{25 \cdot c}{4} = 0{,}05.\right)$$

$$\frac{25c}{4} = 37{,}65 \Rightarrow c = 6{,}024$$

3ème étape

$$S^2 = \frac{1}{n} \cdot \sum (x_i - m)^2 = 4 > c = \cancel{2{,}33}\ 6{,}024.$$

$\Rightarrow RH_0$ Acceptation H_0

tester σ^2 / m inconnu :

$$\begin{cases} H_0 : \sigma^2 = 9 \\ H_1 : \sigma^2 = 7 < 9 \end{cases} \qquad n = 25 \qquad \sum (x_i - \bar{x})^2 = 120.$$

$$RR = \{(x_1 \dots x_n) \mid S'^2 < c\}$$

2ème étape

$$\alpha = P(RH_0 \mid H_0 \text{ vrai})$$
$$= P(S'^2 < c \mid \sigma^2 = 9)$$
$$= P\left(\frac{(n-1)S'^2}{\sigma^2} < \frac{(n-1)c}{\sigma^2} \mid \sigma^2 = 9\right)$$
$$= P\left(\chi^2_{(n-1)} < \frac{24}{9}c\right) = 0{,}05$$
$$\frac{24c}{9} = 14{,}61 \Rightarrow c = 5{,}47$$

Règle de décision

$$S'^2 = \frac{1}{n-1} \sum (x_i - \bar{x})^2 = 5 < c = 5{,}47$$

$\Rightarrow RH_0$

Approximation

Convergence en loi

$X \sim B(n, P) \begin{cases} np < 5 \\ np(1-p) < 15 \end{cases} . X \sim N(np . np(1-p))$

$X \sim B(n,p) \xrightarrow[np<5]{} X \sim P(\lambda = np)$.

$X \sim \chi^2_{(n)} \xrightarrow[n>30]{} X \sim N(n, 2n)$.

$X \sim S(n) \xrightarrow[n>30]{n \to +\infty} X \sim N(0,1)$.

Ex 2.

$X \sim B(\underset{300}{n}, \underset{\frac{6}{60}=\frac{1}{10}}{p})$ X : le nombre d'employés qui tel à un instant (t)

$P(X \geq N) < 0{,}025$

$P(X > N) = 0{,}025$

$X \sim B(np) \xrightarrow[\substack{27 = np(1-p) > 15 \\ np = 30 > 5 \\ n = 300 > 30}]{} X \sim N(\underset{30}{np}, \underset{27}{p(1-p)})$

$P(X < N) = 0{,}975$

$P\left(\frac{X-30}{\sqrt{27}} < \frac{n-30}{\sqrt{27}}\right) = 0{,}975 \Rightarrow F\left(\frac{N-30}{\sqrt{27}}\right) = 0{,}975 \Rightarrow$

$X \sim B(\overset{n}{9000}, \overset{p}{\frac{1}{p}})$

$X \sim B(np) \xrightarrow[\substack{np = 1500 > 5 \\ np(1-p) = 1250 > 15 \\ n = 9000 > 30}]{} X \sim N(\overset{np}{1500}, \overset{np(1-p)}{1250})$

$P(1400 < X < 1600) = P\left(\frac{1400-1500}{\sqrt{1250}} < \frac{X-1500}{\sqrt{1250}} < \frac{1600-1500}{\sqrt{1250}}\right) = P(\quad < Z < \quad)$

Convergence en Probabilité

- X_n Suite de variable aleatoire
  X_n converge en probabilité vers X.

ssi $\lim\limits_{n \to +\infty} (P\, |X_n - X| \geq \varepsilon) = 0.$

On note $X_n \xrightarrow{P} X$

$\overline{X} \xrightarrow{P} E(X)$

$\lim V(X) = E(X - E(X))^2 = 0$

Exercie 1 2018.

$X \sim G(P) \begin{cases} P(X = x) = P(1-P)^{x-1} \\ E(X) = \frac{1}{P} \\ V(X) = \frac{1-P}{P^2} \end{cases}$

$L(x_1 \dots x_n, P) = \prod_{i=1}^{n} P(1-P)^{x_i - 1}.$

$\log L(x_1 \dots x_n, P) = \sum \log (P(1-P)^{x-1})$

$= \sum (\log P + (x-1) \log (1-P))$

$= n \log P + (\sum x_i - n) \log (1-P)$

$\frac{\partial \log (x_1 \dots x_n, P)}{\partial P} = \frac{n}{P} - \frac{\sum x_i - n}{(1-P)} = 0$

$\frac{n(1-P) - P(\sum x_i - n)}{P(1-P)} = 0$

$n - \cancel{nP} - P \sum x_i + \cancel{nP} = 0$

$n = P \sum x_i \Rightarrow \hat{P} = \frac{n}{\sum x_i} = \frac{1}{\frac{\sum x_i}{n}} = \frac{1}{\overline{X}}.$

$\frac{\partial^2 \log (x_1 \dots x_n, P)}{\partial P} = \frac{-n}{P^2} - \frac{(\sum x_i - n) > 0}{(1-P)^2} < 0$ $\quad \hat{P} = \frac{1}{\overline{X}} < 1 \Rightarrow 1 < \overline{X} / \frac{\sum x_i}{n} > 1 / \sum x_i > n$

sans biais: ~~$E(P) = E\left(\frac{1}{\overline{X}}\right) = \frac{1}{E(X)}$~~ $\Rightarrow$ Faux

$E(P) = P = E\left(\frac{1}{\overline{X}}\right)$

$\frac{1}{\overline{X}} \xrightarrow{P} \frac{1}{E(X)} = \frac{1}{\frac{1}{P}} = P$

$\lim V(\hat{P}) = \lim E(\hat{P} - E(\hat{P}))^2 = 0$

$\frac{1}{\overline{X}}\ \hat{P} \xrightarrow{P} E(\hat{P}) = \frac{1}{P}$

l'inégalité de Bienaymé-tchebychev

table KLi: 2 (Student)

Pour tout réel α

$P(|X - E(X)| \geq \alpha) \leq \frac{\sigma^2}{\alpha^2}$.

on jette un dé 3600 fois minorer (mini) la probabilité que le nombre d'apparition du numéro 1 soit compris 480 et 720.

$X \sim B(3600, \frac{1}{6})$

$E(X) = np = 600$.

$V(X) = np(1-p) = 500$.

$480 \leq X \leq 720$ / $-120 \leq X - 600 \leq 120$

l'inégalité de Bienaymé tchebychev

$P(|X - 600| < \underset{\alpha}{120}) \leq \frac{500}{(120)^2}$

$1 - P(|X - 600| > 120) \leq \frac{500}{(120)^2}_{\alpha^2} \Rightarrow P(|X - 600| \geq 120) \geq 1 - \frac{500}{(120)^2} = \alpha^2$